On the Quality of Wireless Network Connectivity

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***Abstract*—Despite intensive research in the area of network connectivity, there is an important category of problems that remain unsolved: how to measure the *quality of connectivity* of a wireless multi-hop network which has a realistic number of nodes, *not necessarily* large enough to warrant the use of asymptotic analysis, and has unreliable connections, reflecting the inherent unreliable characteristics of wireless communications? The quality of connectivity measures how easily and reliably a packet sent by a node can reach another node. It complements the use of *capacity* to measure the quality of a network in saturated traffic scenarios and provides a native measure of the quality of (end-to-end) network connections. In this paper, we explore the use of probabilistic connectivity matrix as a possible tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of connectivity are demonstrated. We argue that the largest eigenvalue of the probabilistic connectivity matrix can serve as a good measure of the quality of network connectivity.**

***Index Terms*—Connectivity, network quality, probabilistic connectivity matrix**

I. Introduction

Connectivity is one of the most fundamental properties of wireless multi-hop networks [1], [2], [3], and is a prerequisite for providing many network functions, e.g. routing, scheduling and localization. A network is said to be *connected* if and only if (iff) there is a (multi-hop) path between any pair of nodes. Further, a network is said to be k-connected iff there are k mutually independent paths between any pair of nodes that do not share any node in common except the starting and the ending nodes. k-connectivity is often required for robust operations of the network.

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There are two general approaches to studying the connectivity problem. The first, spearheaded by the seminal work of Penrose [3] and Gupta and Kumar [1], is based on an asymptotic analysis of large-scale random networks, which considers a network of n nodes that are *i.i.d.* on an area with an underlying uniform distribution. A pair of nodes are directly connected iff their Euclidean distance is smaller than or equal to a given threshold $r(n)$, independent of other connections. Some interesting results are obtained on the value of $r(n)$ required for the above network to be *asymptotically almost surely* connected as $n \to \infty$. In [4], [5], the authors extended the above results by Penrose and Gupta and Kumar from the unit disk model to a random connection model, in which any pair of nodes separated by a displacement $\boldsymbol{x}$ are directly connected with probability $g(\boldsymbol{x})$, independent of other connections. The analytical techniques used in this approach have some intrinsic connections to continuum percolation theory [6] which is usually based on a network setting with nodes Poissonly distributed in an infinite area and studies the conditions required for the network to have a connected component containing an infinite number of nodes (in other words, the network *percolates*). We refer readers to [7] for a more comprehensive review of work in the area.

The second approach is based on a deterministic setting and studies the connectivity and other topological properties of a network using algebraic graph theory. Specifically, consider a network with a set of n nodes. Its property can be studied using its *underlying graph* $G(V,E)$, where $V \triangleq \{v_1, \ldots, v_n\}$ denotes the vertex set and E denotes the edge set. The underlying graph is obtained by representing each node in the network uniquely using a vertex and the converse. An undirected edge exists between two vertices iff there is a direct connection (or link) between the associated nodes[1]. Define an *adjacency matrix* A_G of the graph $G(V,E)$ to be a symmetric $n \times n$ matrix whose $(i,j)^{th}$, $i \neq j$ entry is equal to one if there is an edge between v_i and v_j and is equal to zero otherwise. Further,

[1]In this paper, we limit our discussions to a *simple graph* (network) where there is at most one edge (link) between a pair of vertices (nodes) and an undirected graph.

the diagonal entries of A_G are all equal to zero. The *eigenvalues of the graph* $G(V,E)$ are defined to be the eigenvalues of A_G. The network connectivity information, e.g. connectivity and k-connectivity, is entirely contained in its adjacency matrix. Many interesting connectivity and topological properties of the network can be obtained by investigating the eigenvalues of its underlying graph. For example, let $\mu_1 \geq \ldots \geq \mu_n$ be the eigenvalues of a graph G. If $\mu_1 = \mu_2$, then G is disconnected. If $\mu_1 = -\mu_n$ and G is not empty, then at least one connected component of G is nonempty and bipartite [8, p. 28-6]. If the number of distinct eigenvalues of G is r, then G has a *diameter* of at most $r-1$ [9]. Some researchers have also studied the properties of the underlying graph using its Laplacian matrix [10], where the Laplacian matrix of a graph G is defined as $L_G \triangleq D - A_G$ and D is a diagonal matrix with degrees of vertices in G on the diagonal. Particularly, the *algebraic connectivity* of a graph G is the second-smallest eigenvalue of L_G and it is greater than 0 iff G is a connected graph. We refer readers to [9] and [11] for a comprehensive treatment of the topic. Reference [8] provides a concise summary of major results in the area.

The most related research to the work to be presented in this paper is possibly the more recent work of Brooks et al. [12]. In [12] Brooks *et al.* considered a probabilistic version of the adjacency matrix and defined a *probabilistic adjacency matrix* as a $n \times n$ square matrix M whose $(i,j)^{th}$ entry m_{ij} represents the probability of having a direct connection between distinct nodes i and j, and $m_{ii} = 0$. They established that the probability that there exists at least one walk of length z between nodes i and j is m_{ij}^z, where m_{ij}^z is the $(i,j)^{th}$ entry of $M \otimes M \otimes \cdots \otimes M$ (z times). Here $C \triangleq A \otimes B$ is defined by $C_{ij} = 1 - \prod_{l \neq i,j} (1 - A_{il}B_{lj})$ where A_{ij}, B_{ij} and C_{ij} are the $(i,j)^{th}$ entries of the $n \times n$ square matrix A, B and C respectively. A *walk* of length z between nodes i and j is a sequence of z edges, where the first edge starts at i, the last edge ends at j, and the starting vertex of each intermediate edge is the ending vertex of its preceding edge. A *path* of length z between nodes i and j is a walk of length z in which the edges are distinct.

Despite intensive research in the area, there is an important category of problems that remain unsolved: how to measure the *quality of connectivity* of a wireless multi-hop network which has a realistic number of nodes, *not necessarily* large enough to warrant the use of asymptotic analysis, and has unreliable connections, reflecting the inherent unreliable characteristics of wireless communications? The quality of connectivity measures how easily and reliably a packet sent by a node can reach another node. It complements the use of *capacity* to measure the quality of a network in saturated traffic scenarios and provides a native measure of the quality of (end-to-end) network connections. In the following paragraphs, we elaborate on the above question using using two examples.

Example 1: Consider a network with a fixed number of nodes with known transmission power to be deployed in a certain environment. Assume that the wireless propagation model in that environment is known and its characteristics have been quantified through *a priori* measurements or empirical estimation. Further, a link exists between two nodes iff the received signal strength from one node at the other node, whose propagation follows the wireless propagation model, is greater than or equal to a pre-determined threshold *and* the same is also true in the opposite direction. One can then find the probability that a link exists between two nodes at two fixed locations: It is determined by the probability that the received signal strength is greater than or equal to the pre-determined threshold. Two related questions can be asked: a) If these nodes are deployed at a set of known locations, what is the quality of connectivity of the network, measured by the probability that there is a path between any two nodes, as compared to node deployment at another set of locations? b) How to optimize the node deployment to maximize the quality of connectivity?

Example 2: Consider a network with a fixed number of nodes. The transmission between a pair of nodes with a direct connection, say v_i and v_j, may fail with a known probability, say a_{ij}, quantifying the inherent unreliable characteristics of wireless communications. There are no direct connections between some pairs of nodes because the probability of successful transmission between them is too low to be acceptable. How to measure the quality of connectivity of such a network, in the sense that a packet transmitted from one node can easily and reliably reach another node via a multi-hop path. Will a single "good" path between a pair of nodes be more preferable than multiple "bad" paths? These are further illustrated using Fig. 1 and 2.

In this paper, we explore the use of probabilistic connectivity matrix, a concept to be defined later in Section II, as a possible tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of connectivity are demonstrated.

The rest of the paper is organized as follows. Section II defines the network settings, the probabilistic connectivity matrix and gives a method to compute the matrix. Section III introduces certain inequalities associated with the entries of the probabilistic connectivity matrix. Section IV proves several important results about the probabilistic connectivity matrix. These directly associate the largest eigenvalue of the probabilistic connectivity matrix to the quality of connectivity and expose a structure that holds the promise of facilitating associated optimization tasks. Section V concludes the paper and discusses future work.

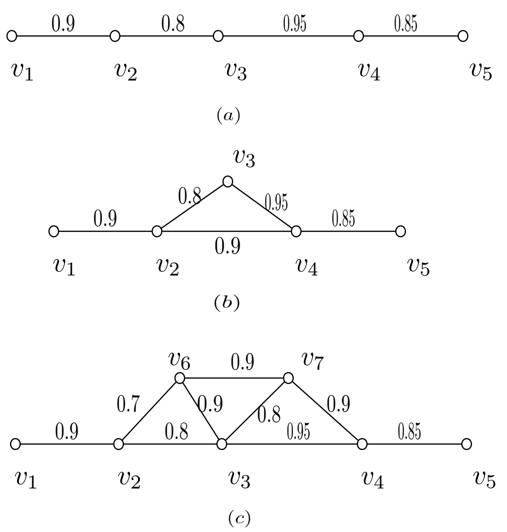

Figure 1. An illustration of networks with different quality of connectivity. A solid line represents a direct connection between two nodes and the number beside the line represents the corresponding transmission successful probability. The networks shown in (a), (b), and (c) are all connected networks but not 2-connected networks, i.e. their connectivity cannot be differentiated using the k-connectivity concept. However intuitively the quality of the network in (b) is better than that of the network in (a) because of the availability of the additional high-quality link between v_2 and v_4 in (b). The quality of the network in (c) is even better because of the availability of the additional nodes and the associated high-quality links, hence additional routes, if these additional nodes act as relay nodes only. If these additional nodes also generate their own traffic, it is uncertain whether the quality of the network in (c) is better or not. Therefore it is important to develop a measure to quantitatively compare the quality of connectivity (for the networks in (a) and (b)) and to evaluate the benefit of additional nodes on connectivity (for the network in (c)).

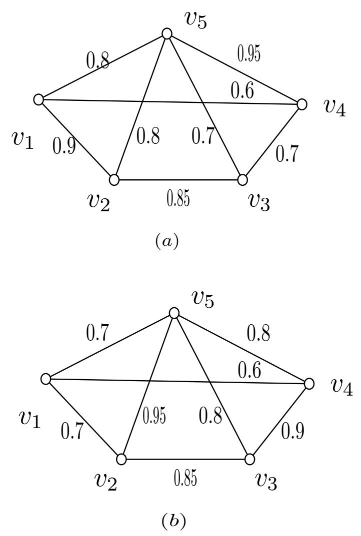

Figure 2. The networks shown in (a) and (b) have the same topology but different link quality. It is difficult to compare the quality of the two networks.

II. Definition and Construction of the Probabilistic Connectivity Matrix

In this section we define the network to be studied, its probabilistic adjacency matrix and probabilistic connectivity matrix, and gives an approach to computing the probabilistic connectivity matrix.

Consider a network of n nodes. For some pair of nodes, an edge (or link) may exist with a non-negligible probability. The edges are considered to be undirected. That is, if a node v_i is connected to a node v_j, then the node v_j is also connected to the node v_i. Further, it is assumed that the event that there is an edge between a pair of nodes and the event that there is an edge between another distinct pair of nodes are independent.

Denote the underlying graph of the above network by $G\left(V,E\right)$, where $V=\{v_1,\ldots,v_n\}$ is the vertex set and $E=\{e_1,\ldots,e_m\}$ is the edge set, which contains the set of *all possible* edges. Here both the vertices and the edges are indexed from 1 to n and from 1 to m respectively. For convenience, in some parts of this paper we also use the symbol e_{ij} to denote an edge between vertices v_i and v_j when there is no confusion. We associate with each edge e_i, $i\in\{1,\ldots m\}$, an indicator random variable I_i such that $I_i=1$ if the edge e_i exists; $I_i=0$ if the edge e_i does not exist. The indicator random variables I_{ij}, $i\neq j$ and $i,j\in\{1,\ldots n\}$, are defined analogously.

In the following, we give a definition of the probabilistic adjacency matrix:

Definition 1: The probabilistic adjacency matrix of $G\left(V,E\right)$, denoted by A_G, is a $n\times n$ matrix such that its $(i,j)^{th}$, $i\neq j$, entry $a_{ij}\triangleq\Pr\left(I_{ij}=1\right)$ and its diagonal entries are all equal to 1.

Due to the undirected property of an edge mentioned above, A_G is a symmetric matrix, i.e. $a_{ij}=a_{ji}$. Note that the diagonal entries of A_G are defined to be 1, which is different from that common in the literature. In [13] we have discussed the implication of this definition in the context of mobile ad-hoc networks. This treatment of the diagonal entries can be associated with the fact that a node in the network can keep a packet until better transmission opportunity arises when it finds the wireless channel busy.

The probabilistic connectivity matrix is defined in the following:

Definition 2: The probabilistic connectivity matrix of $G\left(V,E\right)$, denoted by Q_G, is a $n\times n$ matrix such that its $(i,j)^{th}$, $i\neq j$, entry is the probability that there exists a path between vertices v_i and v_j, and its diagonal entries are all equal to 1.

As a ready consequence of the symmetry of A_G, Q_G is also a symmetric matrix.

Given the probabilistic adjacency matrix A_G, the probabilistic connectivity matrix Q_G is fully determined. However the computation of Q_G is not trivial because for a pair of vertices v_i and v_j, there may be multiple paths between them and some of them may share common edges, i.e. are not *independent*. In the following paragraph, we give an approach to computing the probabilistic connectivity matrix.

Let $(I_1, \ldots, I_m)$ be a particular instance of the indicator random variables associated with an instance of the random edge set. Let $Q_G|\left(I_1, \ldots, I_m\right)$ be the connectivity matrix of G *conditioned on* $(I_1, \ldots, I_m)$. The $(i,j)^{th}$ entry of $Q_G|\left(I_1, \ldots, I_m\right)$ is either 0, when there is no path between v_i and v_j, or 1 when there exists a path between v_i and v_j (see also Lemma 7 in Appendix I). The diagonal entries of $Q_G|\left(I_1, \ldots, I_m\right)$ are always 1. Conditioned on $(I_1, \ldots, I_m)$, $G(V,E)$ is just a deterministic graph. Therefore the entries of $Q_G|\left(I_1, \ldots, I_m\right)$ can be efficiently computed using a search algorithm, such as breadth-first search. Given $Q_G|\left(I_1, \ldots, I_m\right)$, Q_G can be computed using the following equation:

$$Q_G = E\left(Q_G|\left(I_1, \ldots, I_m\right)\right) \tag{1}$$

where the expectation is taken over all possible instances of $(I_1, \ldots, I_m)$.

The approach suggested in the last paragraph is essentially a brute-force approach to computing Q_G. We expect that more efficient algorithms can be designed to compute Q_G. However the main focus of the paper is on exploring the properties of Q_G that facilitate the connectivity analysis and an extensive discussion of the algorithms to compute Q_G is beyond the scope of the paper.

Remark 1: For simplicity, the terms used in our discussion are based on the problems in Example 1. The discussion however can be easily adapted to the analysis of the problems in Example 2. For example, if a_{ij} is defined to be the probability that a transmission between nodes v_i and v_j is successful, the $(i,j)^{th}$ entry of the probabilistic connectivity matrix Q_G computed using (1) then gives the probability that a transmission from v_i to v_j via a multi-hop path is successful under the best routing algorithm, which can always find a shortest and error-free path between from v_i to v_j if it exists, or alternatively, the probability that a packet broadcast from v_i can reach v_j where each node receiving the packet only broadcasts the packet once. Therefore the $(i,j)^{th}$ entry of Q_G can be used as a quality measure of the end-to-end paths between v_i and v_j, which takes into account the fact that availability of an extra path between a pair of nodes can be exploited to improve the probability of successful transmissions.

III. Some Key Inequalities for Connection Probabilities

The entries of the probabilistic connectivity matrix give an intuitive idea about the overall quality of end-to-end paths in a network. In this section, we provide some important inequalities that may facilitate the analysis of the quality of connectivity. Some of these inequalities are exploited in the next section to establish some key properties of the probabilistic connection matrix itself.

We first introduce some concepts and results that are required for the further analysis of the probabilistic connectivity matrix Q_G.

For a random graph with a given set of vertices, a particular event is *increasing* if the event is preserved when more edges are added into the graph. An event is *decreasing* if its complement is increasing.

The following theorems on FKG inequality and BK inequality respectively are used:

Theorem 1: [6, Theorem 1.4] (FKG Inequality) If events A and B are both increasing events or decreasing events depending on the state of finitely many edges, then

$$\Pr\left(A \cap B\right) \geq \Pr\left(A\right)\Pr\left(B\right)$$

Theorem 2: [14], [6, Theorem 1.5] (BK Inequality) If events A and B are both increasing events depending on the state of finitely many edges, then

$$\Pr\left(A \Box B\right) \leq \Pr\left(A\right)\Pr\left(B\right)$$

where for two events A and B, $A \Box B$ denotes the event that there exist two *disjoint* sets of edges such that the first set of edges guarantees the occurrence of A and the second set of edges guarantees the occurrence of B.
There is a recent extension of Theorem 2 to two arbitrary events, i.e. events A and B in Theorem 2 do not have to be increasing events [15].

Denote by ξ_{ij} the event that there is a path between vertices v_i and v_j, $i \neq j$. Denote by ξ_{ikj} the event that there is a path between vertices v_i and v_j *and* that path passes through the third vertex v_k, where $k \in \Gamma_n \backslash \{i,j\}$ and Γ_n is the set of indices of all vertices. Denote by η_{ij} the event that there is an edge between vertices v_i and v_j. Denote by π_{ikj} the event that there is a path between vertices v_i and v_k *and* there is a path between vertices v_k and v_j, where $k \in \Gamma_n \backslash \{i,j\}$. Obviously

$$\pi_{ikj} \Rightarrow \xi_{ij} \tag{2}$$

It can also be shown from the above definitions that

$$\xi_{ij} = \eta_{ij} \cup \left(\cup_{k \neq i,j} \xi_{ikj}\right) \tag{3}$$

Let q_{ij}, $i \neq j$, be the $(i,j)^{th}$ entry of Q_G, i.e. $q_{ij} = \Pr\left(\xi_{ij}\right)$. The following lemma can be readily obtained from the FKG inequality and the above definitions

Lemma 1: For two distinct indices $i,j \in \Gamma_n$ and $\forall k \in \Gamma_n \backslash \{i,j\}$

$$q_{ij} \geq \max_{k \in \Gamma_n \backslash \{i,j\}} q_{ik} q_{kj} \tag{4}$$

Proof: It follows readily from the above definitions that the event ξ_{ij} is an increasing event. Due to (2) and the FKG inequality:

$$\Pr\left(\xi_{ij}\right) \geq \Pr\left(\pi_{ikj}\right) = \Pr\left(\xi_{ik} \cap \xi_{kj}\right) \geq \Pr\left(\xi_{ik}\right)\Pr\left(\xi_{kj}\right) \tag{5}$$

The conclusion follows. ■

Lemma 1 gives a lower bound of q_{ij}. The following lemma gives an upper bound of q_{ij}:

Lemma 2: For two distinct indices $i, j \in \Gamma_n$ and $\forall k \in \Gamma_n \backslash \{i, j\}$,

$$q_{ij} \leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - q_{ik} q_{kj}) \quad (6)$$

where $a_{ij} = \Pr(\eta_{ij})$.

Proof: We will first show that $\xi_{ikj} \Leftrightarrow \xi_{ik} \Box \xi_{kj}$. That is, the occurrence of the event ξ_{ikj} is a sufficient and necessary condition for the occurrence of the event $\xi_{ik} \Box \xi_{kj}$.

Using the definition of ξ_{ikj}, occurrence of ξ_{ikj} means that there is a path between vertices v_i and v_j *and* that path passes through vertex v_k. It follows that there exist a path between vertex i and vertex v_k *and* a path between vertex v_k and vertex v_j *and* the two paths do not have edge(s) in common. Otherwise, it will contradict the definition of ξ_{ikj}, particularly as the definition of a path requires the edges to be distinct. Therefore $\xi_{ikj} \Rightarrow \xi_{ik} \Box \xi_{kj}$. Likewise, $\xi_{ikj} \Leftarrow \xi_{ik} \Box \xi_{kj}$ also follows directly from the definitions of ξ_{ikj}, ξ_{ik}, ξ_{kj} and $\xi_{ik} \Box \xi_{kj}$. Consequently

$$\Pr(\xi_{ikj}) = \Pr(\xi_{ik} \Box \xi_{kj}) \leq \Pr(\xi_{ik}) \Pr(\xi_{kj}) \quad (7)$$

where the inequality is a direct result of the BK inequality.

With a little bit abuse of the terminology, in the following derivations we also use ξ_{ikj} to represent the set of edges that make the event ξ_{ikj} happen, and use η_{ij} to denote the edge between vertices v_i and v_j.

Note that the set of edges $\cup_{k \in \Gamma_n \backslash \{i,j\}} \xi_{ikj}$ does not contain η_{ij}. Therefore using (3) and independence of edges (used in the third step)

$$\begin{aligned} q_{ij} &= \Pr\left(\eta_{ij} \cup \left(\cup_{k \in \Gamma_n \backslash \{i,j\}} \xi_{ikj}\right)\right) \\ &= 1 - \Pr\left(\overline{\eta_{ij}} \cap \left(\overline{\cup_{k \in \Gamma_n \backslash \{i,j\}} \xi_{ikj}}\right)\right) \\ &= 1 - (1 - a_{ij}) \Pr\left(\cap_{k \in \Gamma_n \backslash \{i,j\}} \overline{\xi_{ikj}}\right) \\ &\leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} \Pr\left(\overline{\xi_{ikj}}\right) \quad (8) \\ &= 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - \Pr(\xi_{ikj})) \\ &\leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - q_{ik} q_{kj}) \quad (9) \end{aligned}$$

where in (8), FKG inequality and the obvious fact that $\overline{\xi_{ikj}}$ is a decreasing event are used and the last step results due to (7). ■

When there is no edge between vertices v_i and v_j, which is the generic case, the upper and lower bounds in Lemmas 1 and 2 reduce to

$$\max_{k \in \Gamma_n \backslash \{i,j\}} q_{ik} q_{kj} \leq q_{ij} \leq 1 - \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - q_{ik} q_{kj}) \quad (10)$$

The above inequality sheds insight on how the quality of paths between a pair of vertices is related to the quality of paths between other pairs of vertices. It can be possibly used to determine the most effective way of improving the quality of a particular set of paths by improving the quality of a particular (set of) edge(s), or equivalently what can be reasonably expected from an improvement of a particular edge on the quality of end-to-end paths.

The following lemma further shows that relation among entries of the path matrix Q_G can be further used to derive some topological information of the graph.

Lemma 3: If $q_{ij} = q_{ik} q_{kj}$ for three distinct vertices v_i, v_j and v_k, the vertex set V of the underlying graph $G(V, E)$ can be divided into three non-empty and non-intersecting sub-sets V_1, V_2 and V_3 such that $v_i \in V_1$, $v_j \in V_3$ and $V_2 = \{v_k\}$ *and* any possible path between a vertex in V_1 and a vertex in V_2 must pass through v_k. Further, for any pair of vertices v_l and v_m, where $v_l \in V_1$ and $v_m \in V_3$, $q_{lm} = q_{lk} q_{km}$.

Proof: Using (5) in the second step, it follows that

$$\begin{aligned} q_{ij} &= \Pr((\xi_{ij} \backslash \pi_{ikj}) \cup \pi_{ikj}) \\ &= \Pr(\xi_{ij} \backslash \pi_{ikj}) + \Pr(\pi_{ikj}) \\ &\geq \Pr(\xi_{ij} \backslash \xi_{ikj}) + q_{ik} q_{kj} \end{aligned}$$

Therefore $q_{ij} = q_{ik} q_{kj}$ implies that $\Pr(\xi_{ij} \backslash \pi_{ikj}) = 0$ or equivalently $\xi_{ij} \Leftrightarrow \pi_{ikj}$

Further, $\Pr(\xi_{ij} \backslash \pi_{ikj}) = 0$ implies that a *possible* path (i.e. a path with a non-zero probability) connecting v_i and v_k and a *possible* path connecting v_k and v_j cannot have any edge in common. Otherwise a path from v_i to v_j, bypassing v_k, exists with a non-zero probability which implies $\Pr(\xi_{ij} \backslash \xi_{ikj}) > 0$. The conclusion follows readily that if $q_{ij} = q_{ik} q_{kj}$ for three distinct vertices v_i, v_j and v_k, the vertex set V of the underlying graph $G(V, E)$ can be divided into three non-empty and non-overlapping sub-sets V_1, V_2 and V_3 such that $v_i \in V_1$, $v_j \in V_3$ and $V_2 = \{v_k\}$ *and* a path between a vertex in V_1 and a vertex in V_2, if exists, must pass through v_k.

Further, for any pair of vertices v_l and v_m, where $v_l \in V_1$ and $v_m \in V_3$, it is easily shown that $\Pr(\xi_{lm} \backslash \pi_{lkm}) = 0$. Due to independence of edges and further using the fact that $\Pr(\xi_{lm} \backslash \pi_{lkm}) = 0$, it can be shown that

$$\begin{aligned} \Pr(\xi_{lm}) = \Pr(\pi_{lkm}) &= \Pr(\xi_{lk} \cap \xi_{km}) \\ &= \Pr(\xi_{lk}) \Pr(\xi_{km}) \quad (11) \end{aligned}$$

where (11) results due to the fact that under the condition of $\Pr(\xi_{lm} \backslash \pi_{lkm}) = 0$, a path between vertices v_l and v_k and a path between vertices v_k and v_m cannot possibly have any edge in common. ■

An implication of Lemma 3 is that for any three distinct vertices, v_i, v_j and v_k, if a relationship $q_{ij} = q_{ik} q_{kj}$ holds, vertex v_k must be a *critical* vertex whose removal will render the graph disconnected.

IV. Properties of the Connectivity Matrix

Having established some inequalities obeyed by the entries of the probabilistic connectivity matrix Q_G, we now turn to establishing a measure of the quality of network connectivity. At the core of the development in this section is the following result.

Lemma 4: Each off-diagonal entry of the probabilistic connectivity matrix Q_G is a multiaffine[2] function of a_{ij}, $i \in \{1, \ldots, n\}, j > i$.

Proof: Observe that $a_{ij} = \Pr(\eta_{ij})$ and the events η_{ij}, $i \in \{1, \ldots, n\}, j > i$ are independent. The conclusion in the lemma follows readily from the fact that the event associated with each q_{ij}, i.e. there exists a path between vertices v_i and v_j, is a union of intersections of these events η_{ij}, $i \in \{1, \ldots, n\}, j > i$. ■

Not only does the multiaffine structure facilitate the proof of the main result below, we comment later in Remark 2, on how it is potentially useful for performing some of the optimization tasks inherent in maximizing connectivity.

A very desirable property of Q_G is established below.

Theorem 3: The probabilistic connectivity matrix Q_G, defined for the vertex set $V = \{v_1, \cdots, v_n\}$, is a positive semi-definite matrix. Further, Q_G is positive semi-definite but not positive definite iff there exist distinct $i, j \in \{1, \cdots, n\}$, such that $q_{ij} = 1$.

Proof: See Appendix I. ■

Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of Q_G. Note that $\lambda_1 + \cdots + \lambda_n = n$. As an easy consequence of Theorem 3, $n \geq \lambda_1 \geq 1$ and $1 \geq \lambda_n \geq 0$. In the best case, Q_G is a matrix with all entries equal to 1. Then $\lambda_1 = n$ and $\lambda_2 = \cdots = \lambda_n = 0$. In the worst case, Q_G is an identity matrix. Then $\lambda_1 = \cdots = \lambda_n = 1$. This suggests that λ_1, i.e. the largest eigenvalue of Q_G, can be used as a measure of quality of network connectivity and a larger λ_1 indicates a better quality.

We will make this idea more concrete in the following analysis. We start our discussion with a connected network and then extend to more generic cases. We will call a network *connected* if for all $i, j \in \{1, \cdots, n\}$, $q_{ij} > 0$. Obviously the probabilistic connectivity matrix of a connected network is *irreducible* [16, p. 374] as all the entries of the matrix are non-zero. As a measure of the quality of network connectivity, if the path probabilities q_{ij} increase, the largest eigenvalue of the probabilistic connectivity matrix should also increase. This is formally stated in the following theorem:

Theorem 4: Let $G(V, E)$ and $G'(V, E')$ be the underlying graphs of two connected networks defined on the same vertex set V but with different link probabilities. Let Q_G and $Q_{G'}$ be the probabilistic connectivity matrices of G and G' respectively and let $\lambda_{\max}(Q_G)$ and $\lambda_{\max}(Q_{G'})$ be the largest eigenvalues of Q_G and $Q_{G'}$ respectively. If $Q'_G - Q_G$ is a non-zero, non-negative matrix[3], then

$$\lambda_{\max}(Q_G) < \lambda_{\max}(Q_{G'}) \tag{12}$$

Proof: See Appendix II ■

After having analyzed the situation for connected networks, we now move on to the discussion of disconnected networks and show that the largest eigenvalue of the probabilistic connectivity matrix of a *component*, a concept defined in the next paragraph, provides a good measure of the quality of connectivity of that component. We will analyze two basic situations. Results for more complicated scenarios can be readily obtained from these results and Theorem 4, which applies to a connected network.

If the network is not connected, i.e. some entries of its probabilistic connectivity matrix is 0, it can be easily shown that the network can be decomposed into *disjoint components*. A *component* is a maximal set of vertices where the probability that there is a path between any pair of vertices in the component is greater than zero. Two components are said to be *disjoint* if the probabilities that there is a path between any vertex in the first component and any vertex in the second component are all zeros.

Let the total number of components in the network be k and the number of nodes in the i^{th}, $1 \leq i \leq k$, component be n_i. Further, denote the vertex set of the i^{th} component by V_i and denote the subgraph induced on V_i by G_i. Without loss of generality, we assume that the nodes in the network are properly labeled such that

$$V_i = \{v_{\sum_{j=1}^{i-1} n_j + 1}, \cdots, v_{\sum_{j=1}^{i-1} n_j + n_i}\} \tag{13}$$

Let Q_{G_i} be the probabilistic connectivity matrix of G_i. It follows that the probabilistic connectivity matrix of the network Q_G can be expressed in the form of Q_{G_i}, $1 \leq i \leq k$, as

$$Q_G = \text{ diag } \{Q_{G_1}, \cdots, Q_{G_k}\} \tag{14}$$

and

$$\lambda_{\max}(Q_G) = \max\{\lambda_{\max}(Q_{G_1}), \cdots, \lambda_{\max}(Q_{G_k})\} \tag{15}$$

We consider two basic situations: a) there are increases in some entries of Q_G from non-zero values but such increases do not change the number of components in the network; b) there are increases in some entries of Q_G from zero to non-zero values and such increases reduce the number of components in the network.

Under situation a), the vertex set of each component does not change. Let Q_{G_i} be the probabilistic connectivity matrix of a component whose path probabilities have been increased and let $Q_{G'_i}$ be the probabilistic connectivity matrix of the component after the change in path probabilities. Obviously $Q_{G'_i} - Q_{G_i}$ is a non-zero, non-negative

[2] A multiaffine function is affine in each variable when the other variables are fixed.

[3] A matrix is *non-negative* if all its entries are greater than or equal to 0.

symmetric matrix. It then follows easily from Theorem 4 that $\lambda_{\max}\left(Q_{G'_i}\right) > \lambda_{\max}\left(Q_{G_i}\right)$. Depending on whether $\lambda_{\max}\left(Q_{G'_i}\right)$ is greater than $\lambda_{\max}\left(Q_G\right)$ or not however, the largest eigenvalue of the probabilistic connectivity matrix of the network may or may not increase.

We now move on to evaluate situation b) and consider a simplified scenario where increases in the path probabilities merge two originally disjoint components, denoted by G_i and G_j. The more complicated scenario where increases in the path probabilities join more than two originally disjoint components can be obtained recursively as an extension of the above simplified scenario. Let G' be the underlying graph of the network after increases in path probabilities and let G'_{ij} be the subgraph in G' induced on the vertex set $V_i \cup V_j$. Obviously $Q_{G'_{ij}}$ is an irreducible matrix and the following result can be established.

Lemma 5: Under the above settings,

$$\lambda_{\max}\left(Q_{G'_{ij}}\right) > \lambda_{\max}\left(\text{ diag }\{Q_{G_i}, Q_{G_j}\}\right) \tag{16}$$

The proof of Lemma 5 is omitted due to space limitations.

Thus indeed the largest eigenvalues of the probabilistic connection matrices associated with disjoint components measure the quality of the components connection.

Remark 2: The fact that the largest eigenvalue of the probabilistic connectivity matrix measures connectivity, suggests the following obvious optimization. Modify one or more a_{ij} under suitable constraints to maximize the largest eigenvalue of the probabilistic connectivity matrix. Results in [17] and [18] suggest that the multiaffine dependence of the q_{ij} on the a_{ij} *together with the fact that Q_G is positive semi-definite* promise to facilitate such optimization.

V. Conclusions and Further Work

In this paper we have explored the use of the probabilistic connectivity matrix as a possible tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of network connectivity were demonstrated. Particularly, the off-diagonal entries of the probabilistic connectivity matrix provide a measure of the quality of end-to-end connections and we have also provided theoretical analysis supporting the use of the largest eigenvalue of the probabilistic connectivity matrix as a measure of the quality of overall network connectivity. The analysis focused on the comparison of networks with the same number of nodes. For networks with different number of nodes, the largest eigenvalue of the probabilistic connectivity matrix normalized by the number of nodes may be used as the quality metric.

Inequalities between the entries of the probabilistic connectivity matrix were established. These may provide insights into the correlations between quality of end-to-end connections. Further, the probabilistic connectivity matrix was shown to be a positive semi-definite matrix and its off-diagonal entries are multiaffine functions of link probabilities. These two properties are expected to be very helpful in optimization and robust network design, e.g. determining the link whose quality improvement will result in the maximum gain in network quality, and determining quantitatively the relative criticality of a link to either a particular end-to-end connection or to the entire network.

The results in the paper rely on two main assumptions: the links are symmetric *and* independent. We expect that our analysis can be readily extended such that the first assumption on symmetric links can be removed – in fact the results in Section III do not need this assumption. While in the asymmetric case the probabilistic connectivity matrix is no longer guaranteed to be positive semi-definite, we conjecture that the largest eigenvalue retains its significance. Discarding the second assumption requires more work. However, we are encouraged by the following observation. If we introduce conditional edge probabilities into the mix, then Q_G is still a multiaffine function of the a_{ij} and the conditional probabilities. Thus we still expect all the results in Section IV to hold, though the proof may be non-trivial. In real applications link correlations may arise due to both physical layer correlations and correlations caused by traffic congestion.

Another implicit assumption in the paper is that traffic is uniformly distributed and traffic between every source-destination pair is equally important. If this is not the case, a weighted version of the probabilistic connectivity matrix can be contemplated in which the entries of the matrix are weighted by a measure of the importance of the associated source-destination pairs. It remains to be investigated on whether our results can be extended to a weighted probabilistic connectivity matrix.

Appendix I: Proof of Theorem 3

Let $a_U^{(n)}$ be the vector of a_{ij}, $i \neq j$ (remember that a_{ij} is the probability that there is an edge between v_i and v_j):

$$a_U^{(n)} \triangleq [a_{ij}]_{i=1,j>i}^{n}$$

Also define the set

$$\Pi_U^n = \left\{ a_U^{(n)} \middle| \, 0 \leq a_{ij} \leq 1, i \in \{1, \ldots, n\}, j > i \right\}$$

The corners of the above set are given by:

$$\Pi_{U_c}^n = \left\{ a_U^{(n)} \middle| \, a_{ij} \in \{0, 1\}, i \in \{1, \ldots, n\}, j > i \right\}$$

These corners will play an important role in the subsequent development.

We observe that the positive semi-definiteness of a matrix is known to be a convex property, as defined in [18]. That is, if A and B are positive semi-definite then so is $(1-\alpha) A + \alpha B$, $\forall \alpha \in [0, 1]$. In fact one can say a bit more:

Lemma 6: Consider $n \times n$ matrices $A > 0$ and $B \geq 0$. Then: $(1-\alpha)A + \alpha B > 0$, $\forall \alpha \in [0,1)$

Proof: It is well known that there exists a matrix H, such that $H\left(A + A^T\right)H = I$ and for some diagonal Λ, $H\left(B + B^T\right)H = \Lambda \geq 0$. Then the result follows by noting that $(1-\alpha)I + \alpha\Lambda > 0$, $\forall \alpha \in [0,1)$ ∎

Next we provide a key result that exploits the multiaffine dependence of Q_G on the a_{ij}.

Proposition 1: Suppose Q_G is positive semi-definite for all $a_U^{(n)} \in \Pi_{U_c}^n$. Then it is positive semi-definite for all $a_U^{(n)} \in \Pi_U^n$.

Proof: The combined use of Lemma 4 and Corollary 2.1 of [18] proves the result. ∎

We must next show that Q_G is positive semi-definite for all $a_U^{(n)} \in \Pi_{U_c}^n$. The following lemma is used in the proof of the conclusion that Q_G is positive semi-definite for all $a_U^{(n)} \in \Pi_{U_c}^n$:

Lemma 7: Suppose $a_U^{(n)} \in \Pi_{U_c}^n$, then for all i, j, $q_{ij} \in \{0,1\}$.

Proof: When $a_U^{(n)} \in \Pi_{U_c}^n$, either there is an edge between vertices v_i and v_j surely when $a_{ij} = 1$; or there is no edge between vertices v_i and v_j surely when $a_{ij} = 0$. The graph $G(V,E)$ becomes a deterministic graph. It follows that either there is a path between vertices v_i and v_j surely or there is no path between vertices v_i and v_j surely, i.e. for all i, j, $q_{ij} \in \{0,1\}$. ∎

It can be further shown that the following lemma holds:

Lemma 8: Suppose for some distinct i, j, $q_{ij} = 1$. Then row i and row j of Q_G are identical, as are columns i and j.

Proof: Note that Q_G is a symmetric matrix. Thus it suffices to show that the row property holds. One has

$$q_{ij} = q_{ji} = q_{ii} = q_{jj} = 1 \tag{17}$$

Thus the i^{th} and j^{th} entries of the i^{th} and j^{th} rows are identical. Now consider any k distinct from i and j. Using Lemma 1 and (17):

$$q_{ik} \geq q_{ij}q_{jk} = q_{jk} \text{ and } q_{jk} \geq q_{ij}q_{ik} = q_{ik}$$

It follows that: $q_{jk} = q_{ik}$. ∎

We need one last lemma to complete the proof.

Lemma 9: Define u_m to be a m-vector of all ones. Suppose $a_U^{(n)} \in \Pi_{U_c}^n$. Then for some k, there exist positive integers $m_1, \cdots, m_k$ whose sum is n, and a permutation matrix P, such that:

$$Q_G = P \text{ diag } \left\{u_{m_1}u_{m_1}^T, \ldots, u_{m_k}u_{m_k}^T\right\} P^T \tag{18}$$

Proof: We prove the lemma by induction.

As an easy consequence of Lemma 7, the result clearly holds for $n = 2$ because when $n = 2$, Q_G is either equal to an identity matrix or equal to a matrix of all ones.

Now suppose that the lemma holds for all $n \leq m$. For convenience, we use also Q_n to denote Q_G when $|V| = n$.

When $n = m+1$, consider Q_{m+1}, corresponding to any $a_U^{(n)} \in \Pi_{U_c}^n$. Because of Lemma 7, all entries of Q_{m+1} are in $\{0,1\}$. If $q_{ij} = 0$, $\forall i \neq j$, then the results hold with $m_i = 1$ and $k = n$. Now suppose there exists some distinct i and j for which $q_{ij} = 1$. By symmetrically permuting Q_{m+1}, i.e. by relabeling the nodes if necessary, one can without loss of generality choose $\{i,j\} = \{1,2\}$. Choose m_l to equal the number of ones in the first row of this possibly permuted Q_{m+1}. Through a further symmetric permutation/relabeling if necessary, without loss of generality one has:

$$q_{1j} = q_{j1} = 1, \ \forall j \in \{1, \ldots, m_1\}$$

From Lemma 7:

$$q_{1j} = q_{j1} = 0, \ \forall j \in \{m_1 + 1, \ldots, n\}$$

From Lemma 8, there holds:

$$q_{ij} = \begin{cases} 1 & \forall i,j \in \{1,\ldots,m_1\} \\ 0 & \forall i \in \{1,\ldots,m_1\} \text{ and } j \in \{m_1+1,\ldots,n\} \\ 0 & \forall j \in \{1,\ldots,m_1\} \text{ and } i \in \{m_1+1,\ldots,n\} \end{cases}$$

Thus after relabeling one can express:

$$Q_G = \text{ diag } \left\{u_{m_1}u_{m_1}^T, Q_{m+1-m_1}\right\}$$

Further, there is no path from the vertex set $\{v_1, \ldots, v_{m_1}\}$ to the vertex set $\{v_{m_1}, \ldots, v_n\}$ and vice versa.

Obviously the entries of Q_{m+1-m_1} form legitimate path probabilities with the vertex set $\{v_{m_1}, \ldots, v_n\}$. Then the inductive hypothesis proves the result. ∎

We are now ready to prove Theorem 3.

Observe the matrix in (18) is positive semi-definite. Thus from Lemma 9, Q_G is positive semi-definite for all $a_U^{(n)} \in \Pi_{U_c}^n$. It then follows from Proposition 1 that Q_G is positive semi-definite for all $a_U^{(n)} \in \Pi_U^n$. Therefore the first part of Theorem 3 that Q_G is a positive semi-definite matrix is proved.

Now we proceed to prove the second part of Theorem 3 that Q_G is positive semi-definite but not positive definite iff there exist distinct $i, j \in \{1, \cdots, n\}$, such that $q_{ij} = 1$.

Suppose there exists distinct i and j such that $q_{ij} = 1$. Then from Lemma 8, at least two rows of Q_G are identical. Thus Q_G is singular and cannot be positive definite.

It remains to show that if for some $a_U^{(n)} \in \Pi_U^n$, all $q_{ij} \neq 1$ where $i \neq j$, then Q_G is positive definite. We prove this by induction.

The result is clearly true for $n = 2$. Suppose it holds for some $n = m$. Consider $n = m+1$.

To establish a contradiction suppose there is a $a_U^{(n)} \in \Pi_U^n$ for which all $q_{ij} \neq 1$ where $i \neq j$, and yet Q_n is not positive definite. This means all $a_{ij} \neq 1$ where $i \neq j$.

If for all $i \in \{1, \ldots, n-1\}$, $a_{in} = 0$, then for all $i \in \{1, \ldots, n-1\}$, $q_{in} = 0$. Then vertex v_n is disconnected from the vertex set $\{v_1, \ldots, v_{n-1}\}$ and for all $\{i,j\} \subseteq \{1, \ldots, n-1\}$, q_{ij} are valid path probabilities in the subgraph induced on the vertex set $\{v_1, \ldots, v_{n-1}\}$. Further $Q_n = diag\{Q_{n-1}, 1\}$. By hypothesis Q_{n-1} is positive definite and thus so is Q_n. Because of the resulting contradiction with the hypothesis that Q_n is not positive definite, it follows that for at least one $i \in \{1, \ldots, n-1\}$, $a_{in} \neq 0$.

Through relabeling, without sacrificing generality, assume that $a_{in} \neq 0, \forall i \in \{1, \ldots, l\}$ and $a_{in} = 0, \forall i \in \{l+1, \ldots, n-1\}$.

Define R to be the probabilistic connectivity matrix obtained by keeping all corresponding a_{ij} the same except a_{ln}, which is set to zero. Similarly, define S to be the probabilistic connectivity matrix obtained by keeping all corresponding a_{ij} the same except a_{ln}, which is set to 1.

Observe, because of Lemma 4, that Q_n is a convex combination of R and S. Further, due to the hypothesis that Q_n is not positive definite, both R and S are positive semi-definite but neither can be positive definite. Otherwise, using Lemma 6, Q_n will be positive definite which leads to a contradiction of the hypothesis. In particular R is positive semi-definite, but not positive definite. Now working with R, if $l > 1$, then the probabilistic connectivity matrix obtained by keeping all corresponding a_{ij} the same except a_{2n}, which is set to zero, is similarly positive semi-definite, but not positive definite.

Working recursively in this fashion, the probabilistic connectivity matrix obtained by keeping all a_{ij} the same except $a_{in} = 0$, $\forall i \in \{1, \ldots, n-1\}$, is positive semi-definite, but not positive definite. Thereby a contradiction is established with the conclusion obtained in the earlier paragraph that for at least one $i \in \{1, \ldots, n-1\}$, $a_{in} \neq 0$.

Appendix II: Proof of Theorem 4

The proof of this Theorem appeals to the celebrated Perron-Frobenius theorem whose basics we recount below [16, p. 536].

Theorem 5: Suppose a matrix $A \in \mathbb{R}^{n \times n}$ is non-negative and irreducible. Then the largest eigenvalue of A is simple, positive and has a corresponding eigenvector all whose elements are positive. If A is reducible then the largest eigenvector corresponding to its largest eigenvalue can be chosen to be non-negative.

We also need the following lemma to prove Theorem 4.

Lemma 10: Suppose $A = A^T \neq B = B^T$ are non-negative, irreducible, real matrices, and $B - A$ is a non-zero, non-negative matrix. Then: $\lambda_{\max}(A) < \lambda_{\max}(B)$.

Proof: Observe at least one element of $B - A$ is positive. From Theorem 5, $x \in \mathbb{R}^n$, the eigenvector corresponding to the largest eigenvalue of A can be chosen to have all elements positive. Then the result follows from the fact that:

$$\begin{aligned} \lambda_{\max}(A) x^T x &= x^T A x \\ &= x^T B x - x^T (B - A) x \\ &< x^T B x \\ &\leq \lambda_{\max}(B) x^T x \end{aligned}$$

as $B - A$ is a non-zero, non-negative matrix. ■

Turning to the proof of Theorem 4 we note that the result follows directly from Lemma 10 and the fact $Q_{G'}$ and Q_G satisfy the requirements of B and A, respectively.

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