On the Quality of Wireless Network Connectivity

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***Abstract*—Despite intensive research in the area of network connectivity, there is an important category of problems that remain unsolved: how to measure the *quality of connectivity* of a wireless multi-hop network which has a realistic number of nodes, *not necessarily* large enough to warrant the use of asymptotic analysis, and has unreliable connections, reflecting the inherent unreliable characteristics of wireless communications? The quality of connectivity measures how easily and reliably a packet sent by a node can reach another node. It complements the use of *capacity* to measure the quality of a network in saturated traffic scenarios and provides a native measure of the quality of (end-to-end) network connections. In this paper, we explore the use of probabilistic connectivity matrix as a tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of connectivity are demonstrated. We show that the largest eigenvalue of the probabilistic connectivity matrix can serve as a good measure of the quality of network connectivity.**

***Index Terms*—Connectivity, network quality, probabilistic connectivity matrix**

I. Introduction

Connectivity is one of the most fundamental properties of wireless multi-hop networks [1]–[3], and is a prerequisite for providing many network functions. A network is said to be *connected* if and only if (iff) there is a (multi-hop) path between any pair of nodes. Further, a network is said to be k-connected iff there are k mutually independent paths between any pair of nodes that do not share any node in common except the starting and the ending nodes. k-connectivity is often required for robust operations of the network.

There are two general approaches to studying the connectivity problem. The first, spearheaded by the seminal work of Penrose [3] and Gupta and Kumar [1], is based on an asymptotic analysis of large-scale random networks, which considers a network of n nodes that are *i.i.d.* on an area with an underlying uniform distribution. A pair of nodes are directly connected iff their Euclidean distance is smaller than or equal to a given threshold $r(n)$, independent of other connections. Some interesting results are obtained on the value of $r(n)$ required for the above network to be *asymptotically almost surely* connected as $n \to \infty$. In [4], [5], the authors extended the above results from the unit disk model to a random connection model, in which any pair of nodes separated by a displacement $\boldsymbol{x}$ are directly connected with probability $g(\boldsymbol{x})$, independent of other connections. We refer readers to [7] for a more comprehensive review of related work.

The second approach is based on a deterministic setting and studies the connectivity and other topological properties of a network using algebraic graph theory. Specifically, consider a network with a set of n nodes. Its property can be studied using its *underlying graph* $G(V,E)$, where $V \triangleq \{v_1, \ldots, v_n\}$ denotes the vertex set and E denotes the edge set. The underlying graph is obtained by representing each node in the network uniquely using a vertex and the converse. An undirected edge exists between two vertices iff there is a direct connection (or link) between the associated nodes[1]. Define an *adjacency matrix* A_G of the graph $G(V,E)$ to be a symmetric $n \times n$ matrix whose $(i,j)^{th}, i \neq j$ entry is equal to one if there is an edge between v_i and v_j and is equal to zero otherwise. Further, the diagonal entries of A_G are all equal to zero. The *eigenvalues of the graph* $G(V,E)$ are defined to be the eigenvalues of A_G. The network connectivity information, e.g. connectivity and k-connectivity, is entirely contained in its adjacency matrix. Many interesting connectivity and topological properties of the network can be obtained by investigating the eigenvalues of its underlying graph. For example, let $\mu_1 \geq \ldots \geq \mu_n$ be the eigenvalues of a graph G. If $\mu_1 = \mu_2$, then G is disconnected. If $\mu_1 = -\mu_n$ and G is not empty, then at least one connected component of G is nonempty and bipartite. If the number of distinct eigenvalues of G is r, then G has a *diameter* of at most $r-1$ [8]. Some researchers have also studied the properties of the underlying graph using its Laplacian matrix [9], where the Laplacian matrix of a graph G is defined as $L_G \triangleq D - A_G$ and D is a diagonal matrix with degrees of vertices in G on the diagonal. Particularly, the *algebraic connectivity* of a graph G is the second-smallest eigenvalue of L_G and it is greater than 0 iff G is a connected graph. The algebraic connectivity quantifies the speed of convergence of consensus algorithms [10]. We refer readers to [8] for a comprehensive treatment of the topic.

Despite intensive research in the area, there is an important category of problems that remain unsolved: how to measure the *quality of connectivity* of a wireless multi-hop

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[1]In this paper, we limit our discussions to a *simple graph* (network) where there is at most one edge (link) between a pair of vertices (nodes) and an undirected graph.

network which has a realistic number of nodes, *not necessarily* large enough to warrant the use of asymptotic analysis, and has unreliable connections, reflecting the inherent unreliable characteristics of wireless communications? The quality of connectivity measures how easily and reliably a packet sent by a node can reach another node. It complements the use of *capacity* to measure the quality of a network in saturated traffic scenarios and provides a native measure of the quality of (end-to-end) network connections. In the following paragraphs, we elaborate on the above question using two examples.

Example 1: Consider a network with a fixed number of nodes with known transmission power to be deployed in a region. Assume that the wireless propagation model in that environment is known and its characteristics have been quantified through *a priori* measurements or empirical estimation. Further, a link exists between two nodes iff the received signal strength from one node at the other node is greater than or equal to a predetermined threshold *and* the same is also true in the opposite direction. One can then find the probability that a link exists between two nodes at two fixed locations: It is determined by the probability that the received signal strength is greater than or equal to the pre-determined threshold. Two related questions can be asked: a) If these nodes are deployed at a set of known locations, what is the quality of connectivity of the network, measured by the probability that there is a path between any two nodes, as compared to node deployment at another set of locations? b) How to optimize the node deployment to maximize the quality of connectivity?

Example 2: Consider a network with a fixed number of nodes. The transmission between a pair of nodes with a direct connection quantifying the inherent unreliable characteristics of wireless communications. There are no direct connections between some pairs of nodes because the probability of successful transmission between them is too low to be acceptable. How to measure the quality of connectivity of such a network, in the sense that a packet transmitted from one node can easily and reliably reach another node via a multi-hop path. Will a single "good" path between a pair of nodes be more preferable than multiple "bad" paths? These are further illustrated using Fig. 1 and 2.

In this paper, we explore the use of probabilistic connectivity matrix, a concept to be defined later in Section II, as a tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of connectivity are demonstrated. Based on the analysis, we show that the largest eigenvalue of the probabilistic connectivity matrix can serve as a good metric of the quality of network connectivity.

The rest of the paper is organized as follows. Section II defines the network settings, the probabilistic connectivity matrix and gives a method to compute the matrix. Section III introduces certain inequalities associated with the entries of the probabilistic connectivity matrix. Section IV proves several important results about the probabilistic connectivity matrix. These directly associate the largest eigenvalue of the probabilistic connectivity matrix to the quality of connectivity and expose a structure that holds the promise of facilitating associated optimization tasks. Section V concludes the paper and discusses future work.

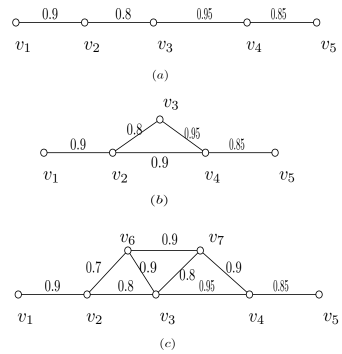

Figure 1: An illustration of networks with different quality of connectivity. A solid line represents a direct connection between two nodes and the number beside the line represents the corresponding transmission successful probability. The networks shown in (a), (b), and (c) are all connected networks but not 2-connected networks, i.e. their connectivity cannot be differentiated using the k-connectivity concept. However intuitively the quality of the network in (b) is better than that of the network in (a) because of the availability of the additional high-quality link between v_2 and v_4 in (b). The quality of the network in (c) is even better because of the availability of the additional nodes and the associated high-quality links, hence additional routes, if these additional nodes act as relay nodes only. If these additional nodes also generate their own traffic, it is uncertain whether the quality of the network in (c) is better or not. Therefore it is important to develop a measure to quantitatively compare the quality of connectivity (for the networks in (a) and (b)) and to evaluate the benefit of additional nodes on connectivity (for the network in (c)).

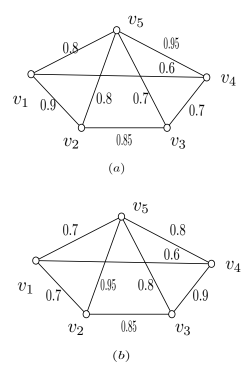

Figure 2: The networks shown in (a) and (b) have the same topology but different link quality. It is difficult to compare the quality of the two networks.

II. Definition and Construction of the Probabilistic Connectivity Matrix

Consider a network of n nodes. For some pair of nodes, an edge (or link) may exist with a non-negligible probability. The edges are undirected and independent.

Denote the underlying graph of the above network by $G(V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the vertex set and $E = \{e_1, \ldots, e_m\}$ is the edge set, which contains the set of *all possible* edges. Here the vertices and the edges are indexed from 1 to n and from 1 to m respectively. For convenience, in some parts of this paper we also use the symbol e_{ij} to denote an edge between vertices v_i and v_j when there is no confusion. We associate with each edge e_i, $i \in \{1, \dots m\}$, an indicator random variable I_i such that $I_i = 1$ if the edge e_i exists; $I_i = 0$ if the edge e_i does not exist. The indicator random variables I_{ij}, $i \neq j$ and $i, j \in \{1, \dots n\}$, are defined analogously.

In the following, we give a definition of the probabilistic adjacency matrix:

Definition 1: The probabilistic adjacency matrix of $G(V, E)$, denoted by A_G, is a $n \times n$ matrix such that its $(i,j)^{th}$, $i \neq j$, entry $a_{ij} \triangleq \Pr(I_{ij} = 1)$ and its diagonal entries are all equal to 1.

Due to the undirected property of an edge mentioned above, A_G is a symmetric matrix, i.e. $a_{ij} = a_{ji}$. Note that the diagonal entries of A_G are defined to be 1, which is different from that common in the literature. This treatment of the diagonal entries can be associated with the fact that a node in the network can store a packet until better transmission opportunity arises when it finds the wireless channel busy [11].

The probabilistic connectivity matrix is defined in the following:

Definition 2: The probabilistic connectivity matrix of $G(V, E)$, denoted by Q_G, is a $n \times n$ matrix such that its $(i,j)^{th}$, $i \neq j$, entry is the probability that there exists a path between vertices v_i and v_j, and its diagonal entries are all equal to 1.

As a ready consequence of the symmetry of A_G, Q_G is also a symmetric matrix.

Given the probabilistic adjacency matrix A_G, the probabilistic connectivity matrix Q_G is fully determined. However the computation of Q_G is not trivial because for a pair of vertices v_i and v_j, there may be multiple paths between them and some of them may share common edges, i.e. are not *independent*. In the following paragraph, we give an approach to computing the probabilistic connectivity matrix.

Let $(I_1, \dots, I_m)$ be a particular instance of the indicator random variables associated with an instance of the random edge set. Let $Q_G|(I_1, \dots, I_m)$ be the connectivity matrix of G *conditioned on* $(I_1, \dots, I_m)$. The $(i,j)^{th}$ entry of $Q_G|(I_1, \dots, I_m)$ is either 0, when there is no path between v_i and v_j, or 1 when there exists a path between v_i and v_j. The diagonal entries of $Q_G|(I_1, \dots, I_m)$ are always 1. Conditioned on $(I_1, \dots, I_m)$, $G(V, E)$ is just a deterministic graph. Therefore the entries of $Q_G|(I_1, \dots, I_m)$ can be efficiently computed using a search algorithm, such as breadth-first search. Given $Q_G|(I_1, \dots, I_m)$, Q_G can be computed using the following equation:

$$Q_G = E\left(Q_G|(I_1, \dots, I_m)\right) \tag{1}$$

where the expectation is taken over all possible instances of $(I_1, \dots, I_m)$.

The approach suggested in the last paragraph is essentially a brute-force approach to computing Q_G. A more efficient algorithm is suggested in Section IV.

Remark 1: A major difference between the (probabilistic) connectivity matrix and the adjacency matrix (or the Laplacian matrix) is that the later matrix focuses on quantifying the relation between node pairs directly connected by an edge only while the former matrix focuses on quantifying the end-to-end relationship between node pairs. It is not trivial to obtain the connectivity matrix from the adjacency matrix or use the adjacency matrix to study network properties easily obtainable using the connectivity matrix.

Remark 2: For simplicity, the terms used in our discussion are based on the problems in Example 1. The discussion however can be easily adapted to the analysis of the problems in Example 2. For example, if a_{ij} is defined to be the probability that a transmission between nodes v_i and v_j is successful, the $(i,j)^{th}$ entry of the probabilistic connectivity matrix Q_G computed using (1) then gives the probability that a transmission from v_i to v_j via a multi-hop path is successful under the best routing algorithm, which can always find a shortest and error-free path from v_i to v_j if it exists, or alternatively, the probability that a packet broadcast from v_i can reach v_j where each node receiving the packet only broadcasts the packet once. Therefore the $(i,j)^{th}$ entry of Q_G can be used as a quality measure of the end-to-end paths between v_i and v_j, which takes into account the fact that availability of extra paths between a pair of nodes can be exploited to improve the probability of successful transmissions.

III. Some Key Inequalities for Connection Probabilities

The entries of the probabilistic connectivity matrix give a measure of the quality of end-to-end paths. In this section, we provide some important inequalities that may facilitate further analysis of the quality of connectivity. Some of these inequalities are exploited in the next section to establish several key properties of the probabilistic connection matrix itself. We first introduce some results that are required for the further analysis of the probabilistic connectivity matrix Q_G.

For a random graph with a given set of vertices, a particular event is *increasing* if the event is preserved when more edges are added into the graph. An event is *decreasing* if its complement is increasing.

Denote by ξ_{ij} the event that there is a path between vertices v_i and v_j, $i \neq j$. Denote by ξ_{ikj} the event that there is a path between vertices v_i and v_j *and* that path passes through the third vertex v_k, where $k \in \Gamma_n \backslash \{i, j\}$ and Γ_n is the set of indices of all vertices. Denote by η_{ij} the event that there is an edge between vertices v_i and v_j. Denote by π_{ikj} the event that there is a path between vertices v_i and v_k *and* there is a path between vertices v_k and v_j, where $k \in \Gamma_n \backslash \{i, j\}$. It can be shown from the above definitions that

$$\xi_{ij} = \eta_{ij} \cup \left(\cup_{k \neq i,j} \xi_{ikj}\right) \tag{2}$$

Let q_{ij}, $i \neq j$, be the $(i,j)^{th}$ entry of Q_G, i.e. $q_{ij} = \Pr(\xi_{ij})$. The following lemma can be readily obtained from the FKG inequality [6, Theorem 1.4] and the above definitions

Lemma 1: For two distinct indices $i, j \in \Gamma_n$ and $\forall k \in \Gamma_n \backslash \{i, j\}$

$$q_{ij} \geq \max_{k \in \Gamma_n \backslash \{i,j\}} q_{ik} q_{kj} \tag{3}$$

Proof: It follows readily from the above definitions that the event ξ_{ij} is an increasing event. Using the FKG inequality:

$$\Pr(\xi_{ij}) \geq \Pr(\pi_{ikj}) = \Pr(\xi_{ik} \cap \xi_{kj}) \geq \Pr(\xi_{ik}) \Pr(\xi_{kj}) \tag{4}$$

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Lemma 1 gives a lower bound of q_{ij}. The following lemma gives an upper bound of q_{ij}:

Lemma 2: For two distinct indices $i, j \in \Gamma_n$ and $\forall k \in \Gamma_n \backslash \{i, j\}$,

$$q_{ij} \leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - q_{ik} q_{kj}) \tag{5}$$

where $a_{ij} = \Pr(\eta_{ij})$.

Proof: We will first show that $\xi_{ikj} \Leftrightarrow \xi_{ik} \Box \xi_{kj}$. That is, the occurrence of the event ξ_{ikj} is a sufficient and necessary condition for the occurrence of the event $\xi_{ik} \Box \xi_{kj}$, where for two events A and B, $A \Box B$ denotes the event that there exist two *disjoint* sets of edges such that the first set of edges guarantees the occurrence of A and the second set of edges guarantees the occurrence of B.

Using the definition of ξ_{ikj}, occurrence of ξ_{ikj} means that there is a path between vertices v_i and v_j *and* that path passes through vertex v_k. It follows that there exist a path between vertex i and vertex v_k *and* a path between vertex v_k and vertex v_j *and* the two paths do not have edge(s) in common. Otherwise, it will contradict the definition of ξ_{ikj}, particularly as the definition of a path requires the edges to be distinct. Therefore $\xi_{ikj} \Rightarrow \xi_{ik} \Box \xi_{kj}$. Likewise, $\xi_{ikj} \Leftarrow \xi_{ik} \Box \xi_{kj}$ also follows directly from the definitions of ξ_{ikj}, ξ_{ik}, ξ_{kj} and $\xi_{ik} \Box \xi_{kj}$. Consequently

$$\Pr(\xi_{ikj}) = \Pr(\xi_{ik} \Box \xi_{kj}) \leq \Pr(\xi_{ik}) \Pr(\xi_{kj}) \tag{6}$$

where the inequality is a direct result of the BK inequality [6] .

With a little bit abuse of the terminology, in the following derivations we also use ξ_{ikj} to represent the set of edges that make the event ξ_{ikj} happen, and use η_{ij} to denote the edge between vertices v_i and v_j.

Note that the set of edges $\cup_{k \in \Gamma_n \backslash \{i,j\}} \xi_{ikj}$ does not contain η_{ij}. Therefore using (2) and independence of edges (used in the third step)

$$\begin{aligned} q_{ij} &= \Pr\left(\eta_{ij} \cup \left(\cup_{k \in \Gamma_n \backslash \{i,j\}} \xi_{ikj}\right)\right) \\ &= 1 - \Pr\left(\overline{\eta_{ij}} \cap \left(\overline{\cup_{k \in \Gamma_n \backslash \{i,j\}} \xi_{ikj}}\right)\right) \\ &= 1 - (1 - a_{ij}) \Pr\left(\cap_{k \in \Gamma_n \backslash \{i,j\}} \overline{\xi_{ikj}}\right) \\ &\leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} \Pr\left(\overline{\xi_{ikj}}\right) \qquad (7) \\ &= 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - \Pr(\xi_{ikj})) \\ &\leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - q_{ik} q_{kj}) \qquad (8) \end{aligned}$$

where in (7), FKG inequality and the fact that $\overline{\xi_{ikj}}$ is a decreasing event are used and the last step results due to (6).

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When there is no edge between vertices v_i and v_j, which is the generic case, the upper and lower bounds in Lemmas 1 and 2 reduce to

$$\max_{k \in \Gamma_n \backslash \{i,j\}} q_{ik} q_{kj} \leq q_{ij} \leq 1 - \prod_{k \in \Gamma_n \backslash \{i,j\}} (1 - q_{ik} q_{kj}) \tag{9}$$

The above inequality sheds insight on how the quality of paths between a pair of vertices is related to the quality of paths between other pairs of vertices. It can be possibly used to determine the most effective way of improving the quality of a particular set of paths by improving the quality of a particular (set of) edge(s), or equivalently what can be reasonably expected from an improvement of a particular edge on the quality of end-to-end paths.

The following lemma further shows that relation among entries of the path matrix Q_G can be further used to derive some topological information of the graph.

Lemma 3: If $q_{ij} = q_{ik} q_{kj}$ for three distinct vertices v_i, v_j and v_k, the vertex set V of the underlying graph $G(V, E)$ can be divided into three non-empty and non-intersecting sub-sets V_1, V_2 and V_3 such that $v_i \in V_1$, $v_j \in V_3$ and $V_2 = \{v_k\}$ *and* any possible path between a vertex in V_1 and a vertex in V_2 must pass through v_k. Further, for any pair of vertices v_l and v_m, where $v_l \in V_1$ and $v_m \in V_3$, $q_{lm} = q_{lk} q_{km}$.

Proof: Using (4) in the second step, it follows that

$$\begin{aligned} q_{ij} = \Pr\left((\xi_{ij} \backslash \pi_{ikj}) \cup \pi_{ikj}\right) &= \Pr(\xi_{ij} \backslash \pi_{ikj}) + \Pr(\pi_{ikj}) \\ &\geq \Pr(\xi_{ij} \backslash \xi_{ikj}) + q_{ik} q_{kj} \end{aligned}$$

Therefore $q_{ij} = q_{ik} q_{kj}$ implies that $\Pr(\xi_{ij} \backslash \pi_{ikj}) = 0$ or equivalently $\xi_{ij} \Leftrightarrow \pi_{ikj}$

Further, $\Pr(\xi_{ij} \backslash \pi_{ikj}) = 0$ implies that a *possible* path (i.e. a path with a non-zero probability) connecting v_i and v_k and a *possible* path connecting v_k and v_j cannot have any edge in common. Otherwise a path from v_i to v_j, bypassing v_k, exists with a non-zero probability which implies $\Pr(\xi_{ij} \backslash \xi_{ikj}) > 0$. The conclusion follows readily that if $q_{ij} = q_{ik} q_{kj}$ for three distinct vertices v_i, v_j and v_k, the vertex set V of the underlying graph $G(V, E)$ can be divided into three non-empty and non-overlapping sub-sets V_1, V_2 and V_3 such that $v_i \in V_1$, $v_j \in V_3$ and $V_2 = \{v_k\}$ *and* a path between a vertex in V_1 and a vertex in V_2, if exists, must pass through v_k.

Further, for any pair of vertices v_l and v_m, where $v_l \in V_1$ and $v_m \in V_3$, it is easily shown that $\Pr(\xi_{lm} \backslash \pi_{lkm}) = 0$. Due to independence of edges and further using the fact that $\Pr(\xi_{lm} \backslash \pi_{lkm}) = 0$, it can be shown that

$$\Pr(\xi_{lm}) = \Pr(\pi_{lkm}) = \Pr(\xi_{lk} \cap \xi_{km}) = \Pr(\xi_{lk}) \Pr(\xi_{km})$$

where the last step results because under the condition of $\Pr(\xi_{lm} \backslash \pi_{lkm}) = 0$, a path between v_l and v_k and a path between v_k and v_m cannot possibly have any edge in common.

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An implication of Lemma 3 is that for any three distinct vertices, v_i, v_j and v_k, if a relationship $q_{ij} = q_{ik} q_{kj}$ holds, vertex v_k must be a *critical* vertex whose removal will render the graph disconnected.

IV. Properties of the Connectivity Matrix

Having established some inequalities obeyed by the entries of Q_G, we now turn to establishing a measure of the quality of network connectivity. At the core of the development in this section is the following result.

Lemma 4: Each off-diagonal entry of the probabilistic connectivity matrix Q_G is a multiaffine[2] function of a_{ij}, $i \in \{1, \dots, n\}, j > i$.

Proof: Observe that $a_{ij} = \Pr(\eta_{ij})$ and the events η_{ij}, $i \in \{1, \dots, n\}, j > i$ are independent. The conclusion in the lemma follows readily from the fact that the event associated with each q_{ij}, i.e. there exists a path between vertices v_i and v_j, is a union of intersections of these events η_{ij}, $i \in \{1, \dots, n\}, j > i$. ∎

Due to the above multiaffine property, for any four positive integers $k, l, i, j \in \{1, \dots n\}$, where $p \neq q$ and $i \neq j$, the following holds:

$$q_{lk} = f\left(E \backslash \{e_{ij}\}\right) a_{ij} + g\left(E \backslash \{e_{ij}\}\right) \quad (10)$$

where $f\left(E \backslash \{e_{ij}\}\right)$ and $g\left(E \backslash \{e_{ij}\}\right)$ are non-negative constants within $[0, 1]$ determined by the state of the set of edges *excluding* e_{ij}. $g\left(E \backslash \{e_{ij}\}\right) = 0$ implies that non-existence of the edge e_{ij} will render the vertices v_l and v_k disconnected. $f\left(E \backslash \{e_{ij}\}\right) = 0$ implies that the state of the edge e_{ij} is irrelevant for the end-to-end paths between v_l and v_k. Further, $f\left(E \backslash \{e_{ij}\}\right)$ can be used to measure the *criticality* of the edge e_{ij} to the end-to-end paths between v_l and v_k.

Remark 3: Using the multiaffine property, a more efficient algorithm for computing Q_G than the one suggested earlier using (1) can be constructed. Particularly, the probabilistic connectivity matrix of a network forming a tree can be easily computed. Therefore the algorithm may start by first identifying a spanning tree in $G(V, E)$ and computing the associated probabilistic connectivity matrix. Then, the edges in E but outside the spanning tree can be added recursively and the corresponding probabilistic connectivity matrix updated using (10).

We comment later in Remark 5 on how the multiaffine structure is also potentially useful for performing some of the optimization tasks inherent in maximizing connectivity. e.g. determination of the link whose improvement will bring the maximum benefit on connectivity.

A very desirable property of Q_G is established below.

Theorem 1: The probabilistic connectivity matrix Q_G is a positive semi-definite matrix. Further, Q_G is positive semi-definite but not positive definite iff there exist distinct $i, j \in \{1, \cdots, n\}$, such that $q_{ij} = 1$.
The proof is omitted due to space limitation.

Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of Q_G. Note that $\lambda_1 + \cdots + \lambda_n = n$. As an easy consequence of Theorem 1, $n \geq \lambda_1 \geq 1$ and $1 \geq \lambda_n \geq 0$. In the best case, Q_G is a matrix with all entries equal to 1. Then $\lambda_1 = n$ and $\lambda_2 = \cdots = \lambda_n = 0$. In the worst case, Q_G is an identity matrix. Then $\lambda_1 = \cdots = \lambda_n = 1$. This suggests that λ_1, i.e. the largest eigenvalue of Q_G, can be used as a measure of quality of network connectivity and a larger λ_1 indicates a better quality.

Further, let X be a vector representing the number of packets broadcast by each node to the rest of the network and let Y be a vector representing the *random* number of packets received by each node. It is obvious that $E[Y|X] = Q_G X$ then represents the expected number of packets received by each node. Using the property that Q_G is a symmetric matrix, it can be shown that

$$\begin{aligned} \max_{||X||_2=1} ||E[Y|X]||_2 &= \max_{||X||_2=1} ||Q_G X||_2 \\ = \max_{||X||_2=1} \sqrt{X^T Q_G^T Q_G X} &= \max_{||X||_2=1} \sqrt{X^T Q_G^2 X} \\ = \sqrt{\lambda_{max}(Q_G^2)} = \sqrt{\lambda_{max}^2(Q_G)} &= \lambda_{max}(Q_G) \end{aligned}$$

where $\lambda_{\max}(Q_G)$ is the maximum eigenvalue of Q_G and $||X||_2$ denotes the L^2-norm or Euclidean norm of X.

We will make this idea that $\lambda_{\max}(Q_G)$ serves as a good measure of the quality of network connectivity more concrete in the following analysis. We start our discussion with a connected network and then extend to more generic cases. We will call a network *connected* if for all $i, j \in \{1, \cdots, n\}$, $q_{ij} > 0$. Obviously the probabilistic connectivity matrix of a connected network is *irreducible* [12, p. 374] as all the entries of the matrix are non-zero. As a measure of the quality of network connectivity, if the path probabilities q_{ij} increase, the largest eigenvalue of the probabilistic connectivity matrix should also increase. This is formally stated below:

Theorem 2: Let $G(V, E)$ and $G'(V, E')$ be the underlying graphs of two connected networks defined on the same vertex set V but with different link probabilities. Let Q_G and $Q_{G'}$ be the probabilistic connectivity matrices of G and G' respectively. If $Q'_G - Q_G$ is a non-zero, non-negative matrix[3], then $\lambda_{\max}(Q_G) < \lambda_{\max}(Q_{G'})$.

Proof:

We need the following lemma to prove Theorem 2.

Lemma 5: Suppose $A = A^T \neq B = B^T$ are non-negative, irreducible, real matrices, and $B - A$ is a non-zero, non-negative matrix. Then: $\lambda_{\max}(A) < \lambda_{\max}(B)$.

Proof: Observe at least one element of $B - A$ is positive. From Perron-Frobenius theorem [12, p. 536], $x \in \mathbb{R}^n$, the eigenvector corresponding to the largest eigenvalue of A can be chosen to have all elements positive. Then the result follows from the fact that:

$$\begin{aligned} \lambda_{\max}(A) x^T x &= x^T A x \\ &= x^T B x - x^T (B - A) x \\ &< x^T B x \leq \lambda_{\max}(B) x^T x \end{aligned}$$

as $B - A$ is a non-zero, non-negative matrix. ∎

Turning to the proof of Theorem 2 we note that the result follows directly from Lemma 5 and the fact $Q_{G'}$ and Q_G satisfy the requirements of B and A, respectively. ∎

If the network is not connected, i.e. some entries of its probabilistic connectivity matrix is 0, the network can be

[2]A multiaffine function is affine in each variable when the other variables are fixed.

[3]A matrix is *non-negative* if all its entries are greater than or equal to 0.

decomposed into *disjoint components*. Let the total number of components in the network be k. Let G_i be the subgraph induced on the set of vertices in the i^{th} component and Q_{G_i} be the probabilistic connectivity matrix of G_i. It follows that

$$\lambda_{\max}(Q_G) = \max\{\lambda_{\max}(Q_{G_1}), \cdots, \lambda_{\max}(Q_{G_k})\} \quad (11)$$

We consider two basic situations: a) there are increases in some entries of Q_G from non-zero values but such increases do not change the number of components in the network. It then follows easily from Theorem 2 that $\lambda_{\max}\left(Q_{G_i'}\right) > \lambda_{\max}(Q_{G_i})$. Depending on whether $\lambda_{\max}\left(Q_{G_i'}\right)$ is greater than $\lambda_{\max}(Q_G)$ or not however, $\lambda_{\max}(Q_G)$ may or may not increase. b) there are increases in some entries of Q_G from zero to non-zero values and such increases reduce the number of components in the network. For situation b), we consider a simplified scenario where increases in the path probabilities merge two originally disjoint components, denoted by G_i and G_j. The more complicated scenario where increases in the path probabilities join more than two originally disjoint components can be obtained recursively as an extension of the above simplified scenario. Let G' be the underlying graph of the network after increases in path probabilities and let G'_{ij} be the subgraph in G' induced on the vertex set $V_i \cup V_j$. Obviously $Q_{G'_{ij}}$ is an irreducible matrix and the following result can be established.

Lemma 6: Under the above settings,

$$\lambda_{\max}\left(Q_{G'_{ij}}\right) > \lambda_{\max}\left(\text{ diag } \{Q_{G_i}, Q_{G_j}\}\right) \quad (12)$$

The proof of Lemma 6 is straightforward and hence omitted.

Thus indeed the largest eigenvalues of the probabilistic connection matrices associated with disjoint components measure the quality of the components connection.

Remark 4: To compare two networks with different number of nodes, the normalized maximum eigenvalue of the probabilistic connectivity matrix, where the maximum eigenvalue is divided by the number of nodes, can be used.

Remark 5: The fact that the largest eigenvalue of the probabilistic connectivity matrix measures connectivity, suggests the following obvious optimization. Modify one or more a_{ij} under suitable constraints to maximize the largest eigenvalue of the probabilistic connectivity matrix. Results in [13] and [14] suggest that the multiaffine dependence of the q_{ij} on the a_{ij} *together with the fact that* Q_G *is positive semi-definite* promise to facilitate such optimization.

V. Conclusions and Further Work

In this paper we explored the use of the probabilistic connectivity matrix as a tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of network connectivity were demonstrated. Particularly, the off-diagonal entries of the probabilistic connectivity matrix provide a measure of the quality of end-to-end connections and we have also provided theoretical analysis supporting the use of the largest eigenvalue of the probabilistic connectivity matrix as a measure of the quality of overall network connectivity.

Inequalities between the entries of the probabilistic connectivity matrix were established. These may provide insights into the correlations between quality of end-to-end connections. Further, the probabilistic connectivity matrix was shown to be a positive semi-definite matrix and its off-diagonal entries are multiaffine functions of link probabilities. These two properties are expected to be very helpful in optimization and robust network design, e.g. determining the link whose quality improvement will result in the maximum gain in network quality, and determining quantitatively the relative criticality of a link to either a particular end-to-end connection or to the entire network.

The results in the paper rely on two main assumptions: the links are symmetric *and* independent. We expect that our analysis can be readily extended such that the first assumption on symmetric links can be removed – in fact the results in Section III do not need this assumption. While in the asymmetric case the probabilistic connectivity matrix is no longer guaranteed to be positive semi-definite, we conjecture that the largest eigenvalue retains its significance. Discarding the second assumption requires more work. However, we are encouraged by the following observation. If we introduce conditional edge probabilities into the mix, then Q_G is still a multiaffine function of the a_{ij} and the conditional probabilities. Thus we still expect all the results in Section IV to hold, though the proof may be non-trivial. In real applications link correlations may arise due to both physical layer correlations and correlations caused by traffic congestion.

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