The Maximum Throughput of A Wireless Multi-Hop Path

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Abstract— **In this paper, the maximum end-to-end throughput that can be achieved on a wireless multi-hop path is investigated analytically. The problem is modeled using the conflict graph, where each link in the multi-hop path is represented uniquely by a vertex in the conflict graph and two vertices are adjacent if and only if the associated links mutually interfere. Using the conflict graph and the linear programming formulations of the problem, we analyzed the maximum end-to-end throughput of a wireless multi-hop path a) in a simple scenario where nodes are optimally placed and each node can only interfere with the transmission of its adjacent nodes along the path, and b) in a more complicated scenario where nodes are randomly placed and each node can interfere with the transmission of any number of nearby nodes along the path in both a) an error free radio environment and b) an erroneous radio environment. The maximum end-to-end throughputs for each of the above four scenarios are obtained analytically. We show that the maximum achievable end-to-end throughput is determined by the throughput of its bottleneck clique, where a clique is a maximal set of mutually adjacent vertices in the associated conflict graph. Further our analysis suggests the optimum scheduling algorithm that can be used to achieve the maximum end-to-end throughput and that it is convenient to use the (maximal) independent sets as the basic blocks for the design of scheduling algorithms. The findings in this paper lay guidelines for the design of optimum scheduling algorithms. They can be used to design computationally efficient algorithms to determine the maximum throughput of a wireless multi-hop path and to design a scheduling algorithm to achieve that throughput.**

Index Terms— **capacity; wireless path; conflict graph; chromatic number**

I. Introduction

Generally, a wireless multi-hop network, e.g. vehicular network, mobile ad-hoc network and wireless sensor network, consists of a group of decentralized and self-organized nodes that communicate with each other over wireless channels, and packets are forwarded hop-by-hop by the wireless nodes collaboratively from the source to the destination without the need for base stations or any fixed infrastructure. The increasing use of wireless multi-hop networks in military and civilian applications demand better understanding on their fundamental properties.

The capacity of wireless multi-hop networks is one of the most fundamental research issues in this area. The most well-known research in this area is by Gupta *et al.* In [1], they analyse the capacity of a random network with a total of n nodes identically, independently, randomly and uniformly distributed on a disk of unit area in $\Re^2$, where any two nodes in the network are directly connected (or adjacent) if and only if their Euclidean distance is smaller than or equal to a given threshold, known as the transmission range, and show that the throughput obtainable by each node for a randomly chosen destination is $\ominus\left(\frac{W}{\sqrt{nlogn}}\right)$ and is $\ominus\left(\frac{W}{\sqrt{n}}\right)$ for optimally placed nodes, where W is the link capacity. Grossglauser *et al.* [2] further demonstrate that in mobile random networks with unbounded delay requirement the per-source-destination throughput decreases approximately like $\frac{1}{\sqrt{n}}$. Compared with the earlier results by Gupta *et al.* [1] for static networks, the results by Grossglauser *et al.* imply that a higher throughput can be achieved in mobile networks, i.e. "mobility increases the capacity of wireless networks".

Different from the work of Gupta *et al.* [1] and Grossglauser *et al.* [2] on analyzing the asymptotic capacity of 2D random wireless network. In this paper, we focus on obtaining an accurate end-to-end capacity for a finite number of wireless nodes, viz. the maximum throughput of a wireless multi-hop path. A wireless multi-hop path constitutes a basic element of wireless multi-hop networks. Investigating the capacity or the maximum throughput of a wireless multi-hop path has both theoretical and practical significance. First, it provides results on the capacity of 1D wireless multi-hop networks. Second, its sheds insight into the analysis of the capacity of higher dimensional networks with multiple sources and destinations. Finally, analysis on the capacity of a wireless multi-hop path are useful for many real world applications which can be modeled by one-dimensional networks. For example, a vehicular network built along a highway can be considered as a 1D network. Other examples of 1D networks include a network deployed along an attack route in battlefield, and a sensor network built for monitoring roads, rivers, coasts and boundaries of restricted areas [3]–[5].

In this paper, we analyze the maximum end-to-end throughput of a wireless multi-hop path, where the source and the destination are placed at both ends of the path, in both an error free and an erroneous radio environment. Compared with previous work in the area, whose results are either validated using simulations or established numerically, e.g. based on

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linear programming (LP) formulations [6]–[11], this paper features a thorough analytical study of the problem. Therefore, this paper offers better insight into the understanding on the maximum throughput of wireless multi-hop networks and the relationship among parameters determining the throughput, which is difficult to obtain by examining simulation results or the outcomes of computing software. For example, for a k-hop ($k \geq 3$) path in an error free radio environment where nodes are optimally placed, our analysis not only gives the maximum throughput, but also demonstrates that this maximum throughput can only be achieved by separating the links into three maximal independent sets, a concept which will be defined later, and scheduling each set of links to be active alternatively for the same amount of time. Further, our analysis shows that in an erroneous radio environment the maximum throughput of the above k-hop path is determined by its bottleneck consecutive three-hop segment. These results are not obvious by examining simulation results or outcomes of computing software.

The main contributions of this paper are: first, we prove that in an error free radio environment, a wireless multi-hop path with at least three hops, in which all nodes are optimally placed and all links have an identical normalized capacity of 1, has a maximum end-to-end throughput of $\frac{1}{3}$. This maximum throughput can only be achieved when all links of the path are separated into three maximal independent sets and each set of links become active alternatively for the same amount of time; second, we extend the above analysis to an erroneous radio environment, and prove that the maximum throughput of a wireless k-hop ($k \geq 3$) path is determined by its bottleneck consecutive three-hop segment. That is, the maximum throughput of the k-hop path is the minimum of the maximum throughputs of all its consecutive three-hop segments; third, we relax the assumption in the above analysis that all nodes are optimally placed and consider the more general setting that nodes are randomly placed. We show that in an error free environment the maximum throughput of a wireless multi-hop path, where all links have an identical normalized capacity of 1, is equal to the reciprocal of the clique number of the associated conflict graph of the path. The clique number is shown to be equal to one plus the maximum forward degree of links in the path. Finally we investigate the maximum throughput of a wireless path with random node placement in an erroneous environment and show that the maximum throughput of the path is determined by the throughput of its bottleneck clique. The findings in this paper lay guidelines for the design of optimum scheduling algorithms. They can be used to design computationally efficient algorithms to determine the maximum throughput of a wireless multi-hop path and to design a scheduling algorithm to achieve that throughput.

The rest of this paper is organized as follows: in Section II, related work is discussed; in Section III, the system model and assumptions used in our analysis is introduced; in Section IV, LP formulation of the problem is introduced and the formulation is further improved using the concept of conflict graphs [7]; the maximum throughput of a wireless k-hop path in an error free environment where nodes are optimally placed is analyzed in Section V; in Section VI, earlier analysis in an error free environment is extended to an erroneous radio environment; Section VII relaxes the assumption on optimal node placement used in sections V and VI, and considers the maximum throughput of a wireless k-hop path where nodes are randomly placed in both erroneous and error-free environments; finally, Section VIII concludes this paper.

II. Related Work

Extensive work has been done in this area, In this section, we limit our discussion only to those closely related [6]–[11].

In [6], Li *et al.* assert that the maximum end-to-end throughput of a wireless multi-hop path is $\frac{1}{n}$ of the capacity of the constituent links where each link has an identical capacity and n is the number of consecutive links whose transmission conflicts with each other. However, they did not prove this assertion and their IEEE 802.11 based simulation study also did not verify this assertion. As will be shown later in the paper, this assertion is incomplete.

In [7], Jain *et al.* consider the maximum throughput that can be supported by a network with a given specific placement of wireless nodes and a specific traffic load. They use the conflict graph to analyze the impact of the interference between adjacent links and formulate a multi-commodity flow problem with constraints derived from the conflict graph to compute the optimal throughput that can be supported between sources and destinations. They show that the problem of finding optimal throughput is NP-hard and present methods for numerically computing upper and lower bounds on the optimal throughput.

In [8], Bohacek *et al.* focus on reducing the computational complexity in the optimal bandwidth allocation problem to maximize the capacity of the network. They propose a Lagrange multiplier theory based technique to the optimal bandwidth allocation problem, which significantly reduces the computational complexity while producing a solution that is indistinguishable from the optimal bandwidth allocation.

In [9], the authors consider wireless link scheduling with power control and SINR constraints. They model the wireless network by a hypergraph and formulate the optimal link scheduling problem as a linear program. Further they show that the SINR-constrained scheduling problem is NP-hard.

In [10], Tang *et al.* consider joint link scheduling and power control problem to maximize the network throughput. They argue that considering maximizing throughput only may lead to a severe bias on bandwidth allocation among links and propose to address the problem by including a new parameter, termed the demand satisfaction factor, to characterize the fairness of bandwidth allocation. The problem is formulated and solved using mixed integer linear programming.

In [11], the authors consider joint rate control and scheduling in multi-hop wireless networks. They propose a dual approach through which the rate control problem and the scheduling problem can be decomposed. The numerical solution obtained not only better utilizes the capacity of the network but also ensures fairness among users.

Different from the above work, which either focus on reducing the computational complexity of the link scheduling problem or presenting another formulation of the link scheduling problem considering more constraints, and the problem has to be solved numerically using computing software, in this paper we aim to obtain an analytical solution to the link scheduling problem which maximizes the network capacity. Using computing software to solve the problem may appear convenient, especially when dealing with complex formulations. However, it is quite difficult to obtain any insight by examining the outcomes of computing software and/or the simulation results. This motivates us to study the throughput of the wireless multi-hop path using an analytic approach.

Finally, we want to point out that the scenario considered in this paper is distinctly different from some other previous work (see [1], [12] and references therein) on analyzing the asymptotic capacity of a 2D wireless network in which a large number of wireless nodes are distributed randomly or arbitrarily in a bounded region. Our focus is on obtaining an accurate end-to-end capacity for a finite number of wireless nodes.

III. SYSTEM MODEL AND ASSUMPTIONS

In this paper, we analyze the maximum end-to-end throughput of a wireless multi-hop path with k links and the source and the destination are placed at both ends of the path. A generic form of the wireless multi-hop path being analyzed is shown in Fig. 1. Each node is numbered sequentially from the source to the destination and the link between node j and node $j+1$ is denoted as L_j. The source node and the destination node are respectively node 1 and node $k+1$.

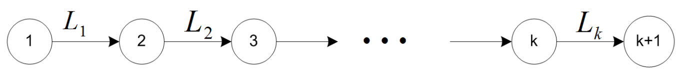


Fig. 1. A generic form of a wireless multi-hop path with k links.

We further assume that:

1) Each node is equipped with a single transceiver and is optimally placed, so that its transmission can only be heard by its adjacent neighbors. Consequently at any time there can only be one link active among three consecutive links. This assumption is only required in Sections V and VI, and is not required in Section VII when we consider a wireless multi-hop path with random node placement.
2) The transmission of an i^{th} node does not cause any interference at the $(i \pm 3)^{th}$ node. This assumption is only necessary when analyzing the throughput in an error-free environment and is no longer required when we consider the throughput in an erroneous environment.
3) The traffic load at the source node of the multi-hop path is saturated. The queue of each node is sufficiently large such that there is no packet loss due to traffic congestion and queue overflow. This assumption is required to establish the maximum end-to-end throughput.
4) All links have an identical normalized capacity of 1. In an erroneous environment, each link has a different and independent successful transmission probability, denoted by P_i $(0 < P_i \leq 1)$. P_i can also be considered as the effective capacity of each link. Thus our results in erroneous case can be readily extended to links with unequal capacities. We would like to further comment that given accurate locations of each node, the path loss model and the physical layer coding technique, the impact of interference can be modeled in the value of $P_i (1 \leq i \leq k)$. The maximum throughput and the associated scheduling algorithm considering the impact of interference can then be computed recursively. In this paper we focus only on the generic properties of wireless multi-hop networks. Such detailed end-to-end throughput evaluation considering very specific scenarios is beyond the scope of the paper.

IV. LP FORMULATIONS AND CONFLICT GRAPH

In this section, we introduce the fundamental principle of our analysis, and define the symbols and terms used in the rest of the paper. First, a LP formulation for calculating the maximum throughput is presented. Second, the concept of conflict graph and its relevance to the throughput analysis are introduced. Finally, the earlier LP formulation is enhanced by using the conflict graph. The improved LP formulation of the maximum end-to-end throughput is used in the later sections to analyze the maximum throughput.

A. A LP formulation for computing the maximum throughput

In this paper, we consider the end-to-end throughput of a wireless multi-hop path with k links from time 0 to t as $t \to \infty$ and the throughput is defined as the time average network traffic transported during this time period. Accordingly, two symbols are defined:

t_j: the proportion of time that link L_j is active during [0, t];

T_j: the throughput of link L_j.

In an error free environment $T_j = t_j$ and in an erroneous environment $T_j = t_j P_j$.

To analyze the maximum throughput of a k-hop path, a LP formulation can be used, given by

$$\text{Maximize:} \quad T_k \tag{1}$$

$$\text{subject to:} \quad 0 \leq t_j \leq 1, j = 1, \ldots, k \tag{2}$$

$$T_j \geq T_{j+1}, j = 1, ..., k-1 \tag{3}$$

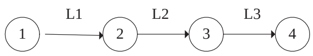
$$t_j + t_{j+1} + t_{j+2} \leq 1, j = 1, ..., k-2. \tag{4}$$

This maximization problem maximizes the end-to-end throughput of the wireless multi-hop path, which is equal to the throughput of the last hop. The first constraint represents that the proportion of time in which a link is active must be between 0 and 1. The second constraint indicates that the throughput of a link cannot exceed that of its previous link. The third constraint indicates that the proportion of the time in which any three consecutive links are active cannot exceed 1 because, as mentioned earlier, at any time only one of the three consecutive links can be active. This constraint is due to assumption 1 in Section III and will be revised in Section VII when we consider the more generic scenario where nodes are randomly placed.

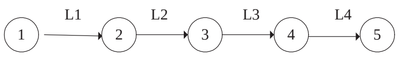
B. Conflict graph

In this section we introduce the concept of conflict graph. A *conflict graph* consists of a number of vertices and edges. Each vertex represents a link. If two links in a multi-hop path can not be active at the same time due to excessive interference, the two links are called *conflicting links* and an edge will be drawn between the corresponding two vertices in the conflict graph. Two links are called *neighbors* (or *adjacent*) if there is an edge between the corresponding vertices in the conflict graph. The *degree* of a link is the number of its neighbors in the conflict graph. In the paper, we use the terms link L_i (in a wireless multi-hop network) and vertex L_i (in a conflict graph) exchangeably; use the term node to refer to a physical node in a wireless network; and use the term edge to refer to a connection between two vertices in a conflict graph. We denote the conflict graph by $G(V, E)$ where V is the vertex set and E is the edge set.

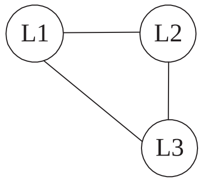
Two examples of the conflict graph are drawn for a three-hop path and a four-hop path respectively, as shown in Fig. 2.



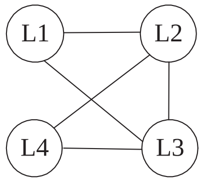
(a) Three-hop path



(b) Four-hop path



(c) Conflict graph of the three-hop path



(d) Conflict graph of the four-hop path

Fig. 2. Examples of conflict graph.

Using the conflict graph, all possible sets of links that can be active simultaneously without interfering each other can be found. These are the sets of links where there is no edge between any two links in the same set. Such a set of links is called an *independent set*. A *maximal independent set* is an independent set that includes as many links as possible. Each independent set is represented by a distinct number and the links in the set. For example, $C_i : \{L_j, L_k\}$ represents the i^{th} independent set and this set includes two links, L_j and L_k. A *maximum independent set* is a maximal independent set that includes the maximum number of links among all maximal independent sets. The number of links (vertices) in a maximum independent set is called the *independence number*, denoted by $\alpha(G)$, of the wireless network (equivalently the conflict graph G).

Using the concept of the conflict graph, a connection can be established between the scheduling problem and the graph coloring problem. Consequently many existing results in graph theory can be applied in the analysis of the maximum throughput and the design of the corresponding scheduling algorithms to achieve that maximum throughput.

A *vertex-coloring* of a graph $G = (V, E)$ is a function

$$\varphi : V \to \mathcal{C} \tag{5}$$

from the set of vertices to a set $\mathcal{C}$ of "colors". A coloring φ is *proper* if no two adjacent vertices are assigned the same color. A graph is k*-colorable* if it admits a proper vertex-coloring with at most k colors. A graph is called k*-chromatic* if it is k-colorable but not $k-1$-colorable. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest nonnegative integer k such that G is k-colorable.

A connection between scheduling problem and graph coloring problem can be established by considering a time slot (or frequency band in FDMA, code in CDMA) as a "color". Given the above association, the minimum number of time slots required in order for all links to be active in at least one time slot without causing any conflict in transmission is $\chi(G)$, the chromatic number of the conflict graph G of the path. Then it is straightforward to show that the maximum end-to-end throughput of the wireless multi-hop path in error-free environment is

$$\frac{1}{\chi(G)}. \tag{6}$$

The chromatic number $\chi(G)$ is related to the independence number by the following inequality [13]

$$\chi(G) \geq \frac{|V|}{\alpha(G)}, \tag{7}$$

where $|V|$ is the cardinality of the vertex set V. A combination of Eq. 6 and Eq. 7 gives an *upper* bound on the maximum end-to-end throughput of a wireless multi-hop path in an error-free environment:

$$\text{Maximum Throughput} = \frac{1}{\chi(G)} \leq \frac{\alpha(G)}{|V|}. \tag{8}$$

However finding the chromatic number or the independence number of a graph is well-known to be NP-hard [14]. In [13], Bollobás shows that the upper bound on the chromatic number in Eq. 7 is almost tight for Erdős-Rényi random graphs [15] but Erdős-Rényi random graphs are not proper models for wireless multi-hop networks because in wireless networks both interference and collision depend on the Euclidean distance between nodes and this fact is not captured in Erdős-Rényi random graphs.

In this paper, we show that under some reasonable assumptions for a wireless multi-hop path, the chromatic number (or the independence number) of the conflict graph can be readily determined and hence the maximum throughput in error-free environments can be readily determined analytically. Further, we investigate the maximum throughput in erroneous environments.

C. An enhanced LP formulation based on independent sets

Using the concept of the conflict graph, we can obtain an enhanced LP formulation of the problem using independent sets as the basic blocks for scheduling.

First, all simultaneously active links constitute one independent set. At any time, at most one independent set is allowed to be active. Note that if two or more independent sets of links are active at the same time, the union of them must form a larger independent set. Thus they are considered as a single independent set.

A new symbol, x_i, is defined which represents the proportion of time that an independent set C_i is active. If C_i includes more than one link, all member links will be active simultaneously.

The value of t_j is related to x_i by

$$t_j = \sum_i \mathbf{1}_{ij} x_i, \tag{9}$$

where $\mathbf{1}_{ij}$ is an indicator function: $\mathbf{1}_{ij} = 1$ if $L_j \in C_i$; otherwise $\mathbf{1}_{ij} = 0$. Note that a link can belong to more than one independent sets. The value of T_j is related to x_i by

$$T_j = \sum_i \mathbf{1}_{ij} x_i P_j. \tag{10}$$

Finally, the LP formulation can be revised as

$$\begin{aligned}
\text{Maximize:} \quad & \textstyle\sum_i \mathbf{1}_{ik} x_i P_k & (11)\\
\text{subject to:} \quad & 0 \le x_i \le 1, i = 1, ... & (12)\\
& \textstyle\sum_i \mathbf{1}_{ij} x_i P_j \ge \sum_i \mathbf{1}_{ij+1} x_i P_{j+1}, j = 1, ..., k-1 & (13)\\
& \textstyle\sum_i x_i \le 1. & (14)
\end{aligned}$$

This LP formation of the maximum throughput problem is valid for any wireless multi-hop path and it does not rely on assumption 1 in Section III. This LP formation of the maximum throughput problem will be used in Section VII when we analyze the maximum throughput of a wireless multi-hop path with random node placement. It shows that it is convenient to use the (maximal) independent sets as the basic blocks for the design of scheduling algorithms.

V. The Maximum Throughput of a k-hop Path with Optimal Node Placement in an Error Free Environment

In this section, we consider the maximum throughput of a wireless k-hop path ($k \ge 3$) with optimal node placement in an error free environment. The main result of this section is summarized in Theorem 1.

Theorem 1: In an error free environment ($P_i = 1$), a wireless k-hop ($k \ge 3$) path where all links have an identical capacity 1 and all nodes are optimally placed, can achieve a maximum throughput of $\frac{1}{3}$. The only way to achieve this maximum throughput is by dividing the links into three maximal independent sets and schedule each set of links to be active alternatively for the same amount of time.

Proof: A mathematical induction approach is used to prove the theorem. First, we prove that Theorem 1 is valid for a three-hop path; second, based on the assumption that the theorem is valid for a $k-1$-hop path, we show that it is also valid for a k-hop path. It is trivial to consider the maximum throughput of a k-hop path with $k < 3$.

1) The conflict graph of a three-hop path is a complete graph. Each independent set of the three-hop path contains a single link only and they are also the maximal independent sets. Therefore, we have $T_1 = t_1 = x_1$, $T_2 = t_2 = x_2$, and $T_3 = t_3 = x_3$, respectively. The maximization of the throughput of the three-hop path can be formulated as the following:

$$\begin{aligned}
\text{Maximize:} \quad & x_3 & (15)\\
\text{subject to:} \quad & 0 \le x_i \le 1, i = 1, 2, 3, & (16)\\
& x_i \ge x_{i+1}, i = 1, 2, & (17)\\
& \textstyle\sum x_i \le 1, i = 1, 2, 3. & (18)
\end{aligned}$$

Rearranging the constraints in the above equations, we can readily obtain three new constraints:

$$\begin{aligned}
0 \le \quad x_2 + x_3 \quad & \le 1, & (19)\\
x_3 \quad & \le \frac{1}{2}(x_1 + x_2), & (20)\\
x_3 \quad & \le 1 - (x_1 + x_2). & (21)
\end{aligned}$$

These three new constraints, together with the constraint in (16), can be used to draw a figure to illustrate the relationship between x_3 and $x_1 + x_2$, as shown in Fig. 3.

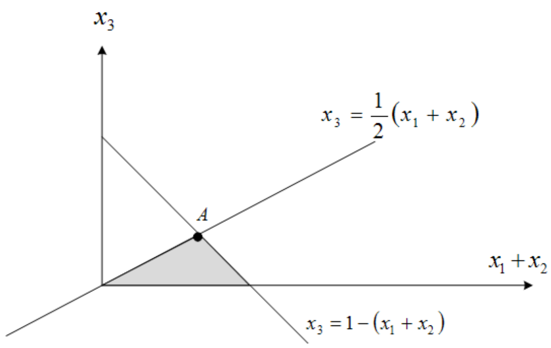


Fig. 3. The relationship between x_3 and $x_1 + x_2$.

In Fig. 3, the vertical axis represents the value of x_3, and the horizontal axis represents the value of $x_1 + x_2$. Two straight lines are drawn from two functions, $x_3 = 0.5(x_1 + x_2)$ and $x_3 = 1 - (x_1 + x_2)$, respectively. Based on the aforementioned constraints, the possible values of x_3 and $x_1 + x_2$ are limited in the shaded triangular area. The only possible values of x_1, x_2 and x_3 which both yield the maximum x_3 and satisfy the constraints are $x_1 = x_2 = x_3 = \frac{1}{3}$. Thus Theorem 1 is proved for the three-hop path.

2) Given that Theorem 1 is valid for a $k-1$-hop path, we shall now show that it is also valid for a k-hop path. Denote the three maximal independent sets of the $k-1$-hop path by C_1, C_2 and C_3. Without losing generality, we consider that the final link L_{k-1} is within C_3. Therefore, when a new node (link), i.e. the $k+1^{th}$ node (k^{th} link), is added to the $k-1$-hop path, the new link L_k together with C_1 forms a maximal independent set of the k-hop path. It is trivial to show that the

three maximal independent sets of the k-hop path are: $\{C_1, L_k\}$, C_2 and C_3.

Obviously, the maximum throughput of the k-hop path is less than or equal to the maximum throughput of the $k-1$-hop path, which is equal to $1/3$. Therefore, if we can show that a throughput of $\frac{1}{3}$ is achievable for the k-hop path, the maximum achievable throughput of the k-hop path is also $\frac{1}{3}$. For the k-hop path, an end-to-end throughput of $\frac{1}{3}$ can be achieved by scheduling L_k to be active at the same time of C_1; and scheduling the three maximal independent sets of the k-hop path: $\{C_1, L_k\}$, C_2 and C_3 to be active alternatively for $1/3$ of the total time.

Next we will show this is also the only way for the k-hop path to achieve a throughput of $1/3$. Given our assumption for the $k-1$-hop path, for the consecutive three-hop segment, $L_{k-3} - L_{k-2} - L_{k-1}$ (note $L_{k-1} \in C_3$, $L_{k-2} \in C_2$ and $L_{k-3} \in C_1$), the proportion of time these three links are active must be chosen such that $t_{k-3} = t_{k-2} = t_{k-1} = \frac{1}{3}$ in order to achieve the maximum throughput of $\frac{1}{3}$ for the $k-1$-hop path. Note that that their sum is 1 and t_k must satisfy $t_k + t_{k-1} + t_{k-2} \leq 1$, therefore link L_k has to be simultaneously active with link L_{k-3} as well as all other links in C_1 in order to achieve the maximum throughput of $1/3$ for the k-hop path and $t_k = 1/3$. These finally lead to the conclusion that Theorem 1 is also valid for the k-hop path.

This part of the proof can also be done by contradiction. ∎

Here we would like to further comment that in the beginning of a packet transmission session, it may happen that although a link in the later hops of the multi-hop path is scheduled to be active, that link may remain silent because there is no packet queued in the corresponding node. We have ignored this effect in our analysis of the maximum throughput of the multi-hop path. As $t \to \infty$, it is expected that the contribution of this effect to the maximum throughput becomes negligibly small.

VI. The Maximum Throughput of a k-hop Path with Optimal Node Placement in an Erroneous Environment

In this section, we extend our analysis in the last section to an erroneous radio environment ($0 < P_i < 1$). First, we obtain the maximum throughput of a generic consecutive three-hop segment. We then extend our analysis to a generic k-hop path.

A. The maximum throughput of a consecutive three-hop segment

Consider the consecutive three-hop segment of the k-hop path, $L_i - L_{i+1} - L_{i+2}$ ($1 \leq i \leq k-2$). Denote the proportion of time that the three links are active by t_i, t_{i+1} and t_{i+2} respectively. The throughputs of these links then become $T_i = P_i t_i$, $T_{i+1} = P_{i+1} t_{i+1}$, and $T_{i+2} = P_{i+2} t_{i+2}$, respectively. Here T_{i+2} is also the throughput of this consecutive three-hop segment. Accordingly, the LP formulation is given by

$$\begin{aligned} \text{Maximize:} \quad & P_{i+2} t_{i+2} && (22) \\ \text{subject to:} \quad & 0 \leq t_j \leq 1, j = i, i+1, i+2 && (23) \\ & t_j P_j \geq t_{j+1} P_{j+1}, j = i, i+1 && (24) \\ & t_i + t_{i+1} + t_{i+2} \leq 1 && (25) \end{aligned}$$

Since P_{i+2} is a known constant, the maximization problem in (22) can be simplified to

$$\text{Maximize: } t_{i+2}. \quad (26)$$

Rearranging the constraints in the above equations (23)-(25), three new constraints on the relationship between t_{i+2} and $t_i + t_{i+1}$ can be obtained, given by

$$\begin{aligned} 0 \leq \quad & t_i + t_{i+1} \quad \leq 1, && (27) \\ t_{i+2} \quad & \leq \quad \frac{t_i + t_{i+1}}{P_{i+2}(\frac{1}{P_i} + \frac{1}{P_{i+1}})}, && (28) \\ t_{i+2} \quad & \leq \quad 1 - (t_i + t_{i+1}). && (29) \end{aligned}$$

A figure is drawn to illustrate the relationship between t_{i+2} and $t_i + t_{i+1}$, as shown in Fig. 4.

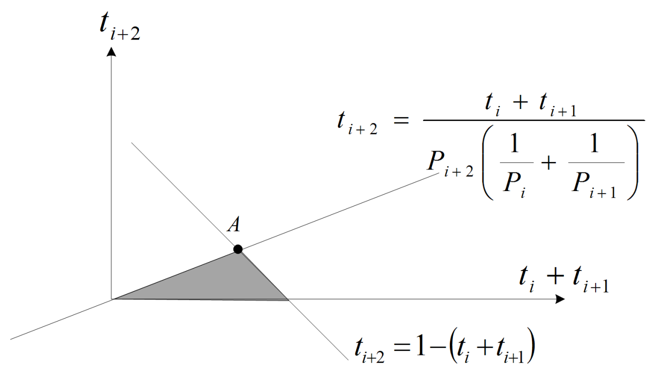


Fig. 4. The relationship between t_{i+2} and $t_i + t_{i+1}$.

Based on Fig. 4, it can be obtained that the maximum value of t_{i+2} while satisfying the constraints is $\frac{1}{P_{i+2}(\frac{1}{P_i} + \frac{1}{P_{i+1}} + \frac{1}{P_{i+2}})}$ and the maximum throughput of the three-hop segment is $\frac{1}{\frac{1}{P_i} + \frac{1}{P_{i+1}} + \frac{1}{P_{i+2}}}$. This maximum throughput is achieved when t_i and t_{i+1} assume the following values

$$t_j = \frac{1}{P_j(\frac{1}{P_i} + \frac{1}{P_{i+1}} + \frac{1}{P_{i+2}})}, j = i, i+1. \quad (30)$$

Note that the values of t_i, t_{i+1} and t_{i+2} corresponding to the maximum throughput satisfy: $t_i + t_{i+1} + t_{i+2} = 1$.

B. The maximum throughput of a k-hop path

Let us now consider the throughput of the generic k-hop path in an erroneous environment and we shall prove the following theorem.

Theorem 2: In an erroneous radio environment, the maximum throughput of a k-hop path ($k \geq 3$) where nodes are optimally placed is the minimum of the maximum throughputs of all consecutive three-hop segment within the k-hop path, that is,

$$\min_{1\le i\le k-2}\frac{1}{\frac{1}{P_i}+\frac{1}{P_{i+1}}+\frac{1}{P_{i+2}}}. \tag{31}$$

Proof: Let us consider that a consecutive three-hop segment within the k-hop path, $L_j - L_{j+1} - L_{j+2}$, is the segment whose maximum throughput is smallest, i.e.

$$\frac{1}{\frac{1}{P_j}+\frac{1}{P_{j+1}}+\frac{1}{P_{j+2}}}=\min_{1\le i\le k-2}\frac{1}{\frac{1}{P_i}+\frac{1}{P_{i+1}}+\frac{1}{P_{i+2}}}. \tag{32}$$

Let us add an adjacent link to the above three-hop segment. Without loss of generality, let us assume the fourth link is L_{j+1}. Obviously the maximum throughput of the four-hop segment $L_j - L_{j+1} - L_{j+2} - L_{j+3}$ cannot exceed that of the three-hop segment $L_j - L_{j+1} - L_{j+2}$. Therefore, if we can show there exists a scheduling algorithm allowing the four-hop segment to achieve a throughput equal the maximum throughput of the three-hop segment, the maximum throughput of the four-hop segment must be the same as the throughput of the three-hop segment. It is trivial to show that a scheduling algorithm with

$$t_l=\frac{1}{P_l(\frac{1}{P_j}+\frac{1}{P_{j+1}}+\frac{1}{P_{j+2}})}, l=j,j+1,j+2,j+3 \tag{33}$$

satisfies our requirements. We will also show that such scheduling algorithm satisfies the constraints in (2)-(4). Let us start with constraint (4). It has been shown earlier that $t_j+t_{j+1}+t_{j+2}=1$. Given (32), it can also be shown that

$$\begin{aligned} t_{j+1}+t_{j+2}+t_{j+3} &= \frac{\frac{1}{\frac{1}{P_{j+1}}+\frac{1}{P_{j+2}}+\frac{1}{P_{j+3}}}}{\frac{1}{P_j}+\frac{1}{P_{j+1}}+\frac{1}{P_{j+2}}} &(34)\\ &\le 1 &(35)\end{aligned}$$

Thus constraint (4) is satisfied. It naturally follows that constraint (2) is also satisfied. Constraint (3) is also satisfied because the scheduling algorithm readily leads to $T_j = T_{j+1} = T_{j+2} = T_{j+3}$. Therefore the maximum throughput of the four-hop segment is also $\frac{1}{\frac{1}{P_j}+\frac{1}{P_{j+1}}+\frac{1}{P_{j+2}}}$.

Repeat the above process and add adjacent links one by one to the consecutive three-hop segment with the minimum maximum throughput, it can be shown that the maximum throughput of a generic k-hop path is equal to the minimum of the maximum throughputs of all consecutive three-hop segments. There exists a scheduling algorithm with

$$t_j=\frac{1}{P_j}\min_{1\le i\le k-2}\frac{1}{\frac{1}{P_i}+\frac{1}{P_{i+1}}+\frac{1}{P_{i+2}}}, 1\le j\le k \tag{36}$$

that allows the k-hop path to achieve its maximum throughput while satisfying the constraints in (2)-(4). ∎

VII. The Maximum Throughput of a k-hop Path with Random Node Placement

In this section, we relax the assumption in Sections V andVI on optimal node placement and consider the more general setting that nodes are randomly placed. Specifically, we consider that the transmission of a link L_i conflicts with that of links $L_{i-m_i}, L_{i-m_i+1}, \ldots, L_{i-1}, L_{i+1}, \ldots, L_{i+n_i-1}, L_{i+n_i}$ where m_i and n_i are nonnegative integers depending on i and may be different for different links. An implication of the above assumption is that if the transmission of link L_i conflicts with the transmission of link L_{i+n_i}, it will also conflict with the transmission of link $L_j, j = i+1,\ldots,i+n_i-1$. This assumption is valid for 1D wireless multi-hop networks and is also a reasonable assumption for most higher dimensional wireless multi-hop networks. Typically the Euclidean distance between links L_i and L_{i+n_i} is larger than the Euclidean distance between links L_i and $L_j, j = i+1,\ldots,i+n_i-1$. Therefore if the transmission of link L_i conflicts with the transmission of link L_{i+n_i}, it will also conflict with the transmission of link $L_j, j = i+1,\ldots,i+n_i-1$. However there may be some special node arrangement in which the assumption is not accurate. Such situations are more likely to occur in low density networks [1] and rarely occur in medium and high density networks. Refer to our paper in [16] for more detailed discussion on the topic. In this paper, we ignore the special situations and consider that the transmission of a link L_i conflicts with that of links $L_{i-m_i}, L_{i-m_i+1}, \ldots, L_{i-1}, L_{i+1}, \ldots, L_{i+n_i-1}, L_{i+n_i}$. Consequently the subgraph of the conflict graph G induced on $L_i, Li+1, \ldots, L_{i+n_i-1}, L_{i+n_i}$ is a complete graph. The *induced subgraph* of a graph $G(V,E)$ on a set of vertices $W \subset V$, denoted $G(W)$, has W as its vertex set and has every edge of G whose endpoints are in W.

We introduce some definitions and concepts that will be used later in the section. We define the *forward degree* of a vertex L_i in the conflict graph as the number of its neighbors whose index is larger than i. Given the assumption in the last paragraph, the forward degree of the vertex L_i is n_i. A *clique* in a graph G is a maximal set of mutually adjacent vertices of G. Obviously an induced subgraph of G on a clique is a complete graph. The *clique number*, denoted by $\omega(G)$, is the number of vertices in a largest clique of G. It can be shown that $\omega(G) = \alpha(\overline{G})$ for any graph G, where $\overline{G}$ denotes the edge-complement graph of G. The *edge-complement graph* $\overline{G}$ has the same vertex set as G. There is an edge between two vertices in $\overline{G}$ if and only if there is no edge between the corresponding vertices in G. It can be shown that the clique number is related to the chromatic number by:

$$\chi(G) \ge \omega(G). \tag{37}$$

Given the assumption in the last paragraph, it is straightforward to show that $\omega(G)-1$ is also equal to the maximum forward degree of the conflict graph, denoted by $\gamma(G)$.

Under the assumption outlined earlier in the section, the following theorem on the maximum throughput of a k-hop path with random node placement in error-free environment can be proved.

Theorem 3: In an error free environment ($P_i = 1$), a k-hop path where all links have an identical capacity 1, can achieve a maximum throughput of $\frac{1}{\omega(G)}$, or equivalently $\frac{1}{\gamma(G)+1}$, where G is the conflict graph of the path. This maximum throughput can be achieved by dividing the links into $\omega(G)$ independent

[1] Most likely the network will be disconnected under such low density.

sets and schedule each set of links to be active alternatively for the same amount of time.

Proof: This theorem can be proved using the same technique used in the proof of Theorem 1. Here we choose to use a different technique based mainly on graph theoretic tools.

First, using Eq. 37 it is straightforward to show that the maximum throughput is not greater than $\frac{1}{\omega(G)}$.

Second, we show that the maximum throughput of $\frac{1}{\omega(G)}$ can be achieved by demonstrating that there exist $\omega(G)$ independent sets whose union equals V where V is the set of vertices in the conflict graph G.

Without loss of generality, assume $L_i, L_{i+1}, \ldots, L_{i+\omega(G)-1}$ is a largest clique of G. Let us start with $\omega(G)$ distinct independent sets where each set contains exactly one vertex in $L_i, L_{i+1}, \ldots, L_{i+\omega(G)-1}$. Now consider adding vertex L_{i-1} into the independence sets. L_{i-1} only conflicts with n_{i-1} vertices in the existing independent sets where $n_{i-1} < \omega(G)$. If $n_{i-1} \geq \omega(G)$, as explained earlier in the section the induced subgraph of G on L_{i-1} and the n_{i-1} vertices conflicting with L_{i-1} becomes a complete graph, which means the clique number of G $\omega(G)$ must satisfy $\omega(G) \geq n_{i-1} + 1 > \omega(G)$. This constitutes a contradiction. Therefore $m_{i-1} < \omega(G)$, which means there exists at least one independent set among the existing $\omega(G)$ independent sets whose elements do not conflict with L_{i-1}. Thus L_{i-1} can be added into one of the existing $\omega(G)$ independent sets. Repeat the above procedure one by one until all vertices on the left side of $L_i, L_{i+1}, \ldots, L_{i+\omega(G)-1}$ are added into the $\omega(G)$ independent sets. Similar procedure can be applied to add vertices on the right side of $L_i, L_{i+1}, \ldots, L_{i+\omega(G)}$ into to the $\omega(G)$ independent sets. Eventually we obtain $\omega(G)$ independent sets whose union is equal to V. As shown in the proof, there are a number of ways to construct the $\omega(G)$ independent sets. Therefore these $\omega(G)$ independent sets are not unique, i.e. there is more than one way to schedule the links to achieve the maximum throughput. ∎

It can be shown that Theorem 1 actually constitute a special case of Theorem 3 where $\omega(G) = 3$ and $\gamma(G) = 2$. The reason for including $\gamma(G)$ in Theorem 3 is that the maximum forward degree of the conflict graph is easier to compute than the clique number. Therefore the inclusion of $\gamma(G)$ provides an easier way to compute the maximum throughput. Not surprisingly, Theorem 3 suggests that the maximum throughput of a wireless multi-hop path is not necessarily equal to the reciprocal of the number of conflicting links, as asserted by some researchers.

Let us now consider the maximum throughput of the k-hop path in an erroneous environment. We shall prove the following theorem:

Theorem 4: In an erroneous radio environment, the maximum throughput of a k-hop path is the minimum of the maximum throughputs of all cliques of its conflict graph, that is,

$$\min_{1 \leq i \leq N} \frac{1}{\frac{1}{P_{M_i}} + \frac{1}{P_{M_i+1}} + \ldots + + \frac{1}{P_{M_i+N_i-1}}}. \tag{38}$$

where N is the total number of cliques, M_i is the vertex with the smallest index in the i-th clique and N_i is the number of vertices in the i-th clique.

Proof:

Note that under the assumption in this section, the vertices in a clique are consecutively indexed.

We prove the above theorem in two steps. First we show that the maximum throughput of the i-th clique is given by:

$$\mathcal{T}_i = \frac{1}{\frac{1}{P_{M_i}} + \frac{1}{P_{M_i+1}} + \ldots + + \frac{1}{P_{M_i+N_i-1}}}. \tag{39}$$

Second, we show that the maximum throughput of the k-hop path is given by Eq. 38.

We prove the first claim by contradiction. Note that links in the same clique cannot be active at the same time. Assume there exists a scheduling algorithm that allows the i-th clique to achieve a maximum throughput greater than that given by Eq. 39. This implies there exists a scheduling algorithm that allows links $L_{M_i}, L_{M_i+1}, \ldots, L_{M_1+N_i-1}$ to be active $t_{M_i}, t_{M_i+1}, \ldots, t_{M_1+N_i-1}$ proportions of time such that

$$P_{M_i} t_{M_i} \geq \cdots \geq P_{M_i+N_i-1} t_{M_i+N_i-1} > \mathcal{T}_i. \tag{40}$$

The total proportion of time these links are active is then

$$\sum_{j=M_i}^{M_i+N_i-1} t_j > \sum_{j=M_i}^{M_i+N_i-1} \frac{1}{P_j} \mathcal{T}_i = 1. \tag{41}$$

This violates the constraint in 14, which implies that links in the same clique cannot be active at the same time. Therefore the maximum throughput of the i-th clique cannot be greater than $\mathcal{T}_i$. Further it is straightforward to show that a scheduling algorithm that allocates a link L_j in $L_{M_i}, L_{M_i+1}, \ldots, L_{M_1+N_i-1}$ a proportion of time $t_j = \frac{\mathcal{T}_i}{P_j}$ is able to achieve the maximum throughput $\mathcal{T}_i$ (Refer to the proof of Theorem 2 for details. Here the same technique can be used.). Therefore the maximum throughput of the i-th clique must be $\mathcal{T}_i$.

Now we consider the end-to-end throughput of the k-hop path with N cliques. It is obvious that the maximum throughput of the k-hop path cannot be greater than the maximum throughout of any of the cliques of its conflict graph, i.e. must be smaller than or equal to the minimum of the maximum throughputs of all cliques of its conflict graph. Assume the i-th clique has the minimum maximum throughput among all cliques, then

$$\mathcal{T}_i = \min_{1 \leq i \leq N} \frac{1}{\frac{1}{P_{M_i}} + \frac{1}{P_{M_i+1}} + \ldots + + \frac{1}{P_{M_i+N_i-1}}}. \tag{42}$$

Using the same technique as that used in the proof of Theorem 2, it can be shown that a scheduling algorithm that allocates a link L_j a proportion of time equal to $\frac{\mathcal{T}_i}{P_j}$ is able to allow the k-hop path to achieve the throughput $\mathcal{T}_i$ while satisfying all constraints in 12, 13 and 14. ∎

VIII. Conclusion

In this paper, we investigated the maximum throughput of a wireless multi-hop path analytically.

We showed that in an ideal error free radio environment, a wireless multi-hop path, in which all nodes are optimally placed and all links has an identical normalized capacity of 1, has a maximum throughput of $\frac{1}{3}$. This maximum throughput can only be achieved when all links of the path are separated into three maximal independent sets and each set of links become active alternatively for the same proportion of time. In an erroneous radio environment, we showed that the maximum throughput of the k-hop path is determined by its bottleneck consecutive three-hop segment. A scheduling algorithm achieving this maximum throughput was also presented.

For a wireless multi-hop path where nodes are randomly placed and all links have a normalized capacity of 1, we showed that in an error-free environment the maximum throughput is equal to the reciprocal of the clique number of its conflict graph; in an erroneous environment the maximum throughput is determined by the bottleneck clique. The results suggest that when analyzing the maximum throughput, cliques should be used as basic units for determining the throughput. The results in this paper will help to design guidelines for optimum scheduling algorithms.

This paper suggests that in order to achieve the maximum throughput, it is necessary to use independence sets as the basic units in the design of scheduling algorithms and synchronize the transmission of links in the same independent set. As these links in the same independence set may belong to different cliques and are located far away, it remains to be investigated as to how these design principles can be realized in a distributed scheduling algorithm.

Finally, the paper considers only a single channel environment. It is an interesting area to investigate the maximum throughput in a multi-channel environment and the optimum scheduling algorithms to achieve that maximum throughput.

REFERENCES

[1] P. R. Gupta and P. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 388–404, 2000.

[2] M. Grossglauser and D. N. C. Kumar, "Mobility increases the capacity of ad hoc wireless networks," *IIEEE/ACM Transactions on Networking*, vol. 10, pp. 477–486, 2002.

[3] C. H. Foh and B. S. Lee, "A closed form network connectivity formula for one-dimensional manets," in *IEEE International Conference on Communications (ICC)*, 2004, p. 3739 3742.

[4] G. J. N. Antunes and A. Pacheco, " On the minimum hop count and connectivity in one-dimensional ad hoc wireless networks ," *Telecommunication Systems*, vol. 39, no. 2, p. 137143, 2008.

[5] L. M. V. Kurlin and S. Maskell, " How many randomly distributed wireless sensors are enough to make a 1-dimensional network connected with a given probability ," *submitted to IEEE Transactions on Information Theory*, 2008, coRR, vol. abs/0710.1001.

[6] J. Li, C. Blake, D. S. J. De Couto, H. I. Lee, and R. Morris, "Capacity of ad hoc wireless networks," in *Proceedings of the 7th ACM International Conference on Mobile Computing and Networking*, Rome, Italy, July 2001, pp. 61–69.

[7] K. Jain, J. Padhye, V. N. Padmanabhan, and L. Qiu, "Impact of interference on multi-hop wireless network performance," in *MobiCom '03: Proceedings of the 9th annual international conference on Mobile computing and networking*. New York, NY, USA: ACM Press, 2003, pp. 66–80.

[8] S. Bohacek and P. Wang, "Toward Tractable Computation of the Capacity of Multi-Hop Wireless Networks," in *IEEE INFOCOM 2007*, 2007, pp. 2099–2107.

[9] S. A. Borbash and A. Ephremides, "Wireless Link Scheduling With Power Control and SINR Constraints," *IEEE Transactions on Information Theory*, vol. 52, no. 11, pp. 5106–5111, 2006, 0018-9448.

[10] J. Tang, G. Xue, C. Chandler, and W. Zhang, "Link scheduling with power control for throughput enhancement in multihop wireless networks," *IEEE Transactions on Vehicular Technology*, vol. 55, no. 3, pp. 733–742, 2006, 0018-9545.

[11] X. Lin and N. B. Shroff, "Joint rate control and scheduling in multihop wireless networks," in *43rd IEEE Conference on Decision and Control, 2004*, vol. 2, 2004, pp. 1484–1489.

[12] F. Massimo, D. Olivier, N. C. T. David, and T. Patrick, "Closing the gap in the capacity of wireless networks via percolation theory," *IEEE Transactions on Information Theory*, vol. 53, no. 3, pp. 1009–1018, 2007.

[13] B. Bollobás, " The Chromatic Number of Random Graphs ," *Combinatorica*, vol. 8, pp. 49–55, 1988.

[14] R. M. Karp, "Reducibility among combinatorial problem," in *Complexity of Computer Computations*.

[15] B. Bollobas, *Modern Graph Theory*, 2nd ed. Springer, 2002.

[16] G. M. X. Ta and B. D. O. Anderson, "Evaluation of the probability of k-hop connection in homogeneous wireless sensor networks," in *IEEE Globecom*, 2008, pp. 1279 – 1284.