A Capacity Upper Bound for Large Wireless Networks with Generally Distributed Nodes

Guoqiang Mao, Zihuai Lin
School of Electrical and Information Engineering
The University of Sydney
Email: {guoqiang.mao, zihuai.lin}@sydney.edu.au

Wei Zhang
School of Electrical Engineering and Telecommunications
The University of New South Wales
Email: w.zhang@unsw.edu.au

***Abstract*—Since the seminal work of Gupta and Kumar, extensive research has been done on studying the capacity of large wireless networks under various scenarios. Most of the existing work focuses on studying the capacity of networks with uniformly or Poissonly distributed nodes. While uniform and Poisson distribution form an important class of spatial distributions, their capability in capturing the spatial distribution of users in various scenarios and application settings is limited. Therefore it is critical to investigate to what extent, the aforementioned results on capacity of networks with uniformly or Poissonly distributed nodes depend on the underlying node distribution being uniform or Poisson. In this paper, we study the capacity of networks under a general node distribution. A capacity upper bound on networks with generally distributed nodes is obtained, which is valid for both finite networks and asymptotically infinite networks. By imposing some mild conditions on the transmission range, we further simplify the result and show that the asymptotic capacity upper bound can be expressed as a product of four factors, which represents respectively the impact of node distribution, link capacity, number of source destination pairs and the transmission range. The upper bound is shown to be tight in the sense that for the special case of networks with uniformly distributed nodes, the bound is in the same order as known results in the literature.**

***Index Terms*—Capacity, node distribution, wireless networks**

I. INTRODUCTION

Since the seminal work of Gupta and Kumar [1], extensive research has been done on studying the capacity of large wireless networks under various scenarios [1]–[9]. More specifically, in [1], Gupta and Kumar considered an ad-hoc network with a total of n nodes uniformly and *i.i.d.* on an area of unit size. Each node in the network is capable of transmitting at W bits/s and using a fixed and identical transmission range. It was shown that when each node randomly and independently chooses another node in the network as its destination, the transport capacity and the achievable per-node throughput are $\Theta_n\left(W\sqrt{\frac{n}{\log n}}\right)$ and $\Theta_n\left(\frac{W}{\sqrt{n\log n}}\right)$ respectively[1]. When the nodes are optimally and deterministically placed to maximize throughput, the transport capacity and the achievable per-node throughput become $\Theta_n\left(W\sqrt{n}\right)$ and $\Theta_n\left(\frac{W}{\sqrt{n}}\right)$ respectively. In [2], Franceschetti *et al.* considered essentially the same random network as that in [1] except that nodes are now allowed to use two different transmission ranges. The link capacity is determined by the associated SINR through the Shannon–Hartley theorem. By having each source-destination pair transmitting via the so-called "highway system", formed by nodes using the smaller transmission range, it was shown in [2] that the transport capacity and the per-node throughput can also reach $\Theta_n\left(\sqrt{n}\right)$ and $\Theta_n\left(\frac{1}{\sqrt{n}}\right)$ respectively even when nodes are randomly deployed. The existence of such highways was analytically proved using the percolation theory [10]. The key to achieving a higher capacity in the network considered in [2] is that nodes are restricted to use the smaller transmission range as often as possible and the larger transmission range can only be used by source (destination) nodes to access their respective nearest highway nodes. In this way, the number of concurrent transmissions that can be accommodated in the network area is maximized, hence the improvement in capacity. In [4] Grossglauser and Tse showed that in mobile ad hoc networks, by leveraging on the nodes' mobility, a per-node throughput of $\Theta_n\left(1\right)$ can be achieved at the expense of unbounded delay. Their work [4] has sparked huge interest in studying the capacity-delay tradeoffs in mobile networks assuming various mobility models and the obtained results often vary greatly with the different mobility models being considered, see [3], [5], [11]–[14] and references therein for examples. In [7], Chen et al. studied the capacity of wireless networks under a different traffic distribution. More specifically, they considered a network with a set of n randomly deployed nodes transmitting to single sink or multiple sinks where the sinks can be either regularly-deployed or randomly-deployed. Under the above settings, it was shown that with single sink, the transport capacity is given by $\Theta_n\left(W\right)$; with k sinks, the transport capacity is increased to $\Theta_n\left(kW\right)$ when $k = O_n(n\log n)$ or $\Theta_n\left(n\log nW\right)$ when $k = \Omega_n\left(n\log n\right)$. There is also a significant amount of work studying the impact of infrastructure nodes [6] and multiple-access protocols [9]

This research is supported by ARC Discovery projects DP110100538 and DP120102030.

[1]The following notations are used throughout the paper. For two positive functions $f\left(x\right)$ and $h\left(x\right)$:

- $f\left(x\right) = o_x\left(h\left(x\right)\right)$ iff (if and only if) $\lim_{x\to\infty}\frac{f(x)}{h(x)} = 0$;
- $f\left(x\right) = \omega_x\left(h\left(x\right)\right)$ iff $h\left(x\right) = o_x\left(f\left(x\right)\right)$;
- $f\left(x\right) = \Theta_x\left(h\left(x\right)\right)$ iff there exist a sufficiently large x_0 and two positive constants c_1 and c_2 such that for any $x > x_0$, $c_1 h\left(x\right) \geq f\left(x\right) \geq c_2 h\left(x\right)$;
- $f\left(x\right) \sim_x h\left(x\right)$ iff $\lim_{x\to\infty}\frac{f(x)}{h(x)} = 1$;
- An event ξ_x depending on x is said to occur asymptotically almost surely (a.a.s.) if its probability tends to one as $x\to\infty$.

on capacity and the multicast capacity [8]. We refer readers to [15] for a comprehensive review of related work.

The above work studying the capacity of random networks has all assumed that nodes in the network are either uniformly or Poissonly distributed. While uniform and Poisson distribution form an important class of spatial distributions and have been extensively used in the area, their capability in capturing the spatial distribution of users in various scenarios and application settings is limited. Therefore it is crucial to investigate to what extent, the aforementioned results on network capacity depend on the underlying node distribution being uniform or Poisson. It is worth noting that in a series of papers [16]–[18], Alfano and Garetto *et al.* studied the capacity of a class of clustered networks in which nodes are distributed according to a doubly stochastic shot-noise Cox process. A doubly stochastic shot-noise Cox process is formed by first deploying a set of nodes, termed cluster centers, randomly and independently in the network area, and then each cluster center generating independently a point process of nodes with a designated density function. The overall node process is then given by the superposition of the individual processes generated by the cluster centers. Their work [16]–[18] generated interesting insight on the performance impact of doubly stochastic shot-noise Coxs process, compared with commonly used Poisson or uniform spatial node distribution. Different from the work of Alfano and Garetto et al. [16]–[18], in this paper we study the capacity of networks under a more general node distribution, to be defined in Section II.

In addition to capacity, delay is also an important performance metric that has been extensively investigated. In this paper we focus on the study of capacity. We refer readers to [3], [5], [11]–[13] for relevant work on delay.

The following is a detailed summary of our contributions:

- We develop a novel method of analyzing information exchanges across a closed curve for studying the capacity of large wireless networks, distinct from the methods used in previous work following the methodology in [1];
- Using the method, we derive the capacity upper bound of networks with generally distributed nodes. The capacity upper bound is valid for both finite and infinite networks. The method is shown to be effective and efficient for analyzing the capacity of large networks in a more general setting;
- By imposing some mild conditions on the transmission range, we simplify the capacity upper bound to gain insight on the interactions among major capacity-impacting factors. We show that the (asymptotic) capacity upper bound of networks with generally distributed nodes is determined by four factors, i.e. node distribution, the link capacity, the number of source-destination pairs and the transmission range, in a multiplicative form. The tightness of the upper bound is validated by comparing the upper bound with known results obtained assuming uniform or Poisson node distribution.

The rest of this paper is organized as follows: Section II presents the network model of interest; Section III presents theoretical analysis on the capacity upper bound of networks with generally distributed nodes; in Section IV, by imposing some mild conditions on the transmission range, we simplify the results obtained in Section III and discuss insight revealed through the simplified results. The tightness of the capacity upper bound is also validated. Finally, Section V concludes this paper.

II. Network Model

Two network models are widely used in the study of (asymptotic) network capacity: the *dense network model* and the *extended network model*. By appropriate scaling of the distance, the results obtained under one model are often readily extendable to the other model [19]. In this paper, we consider the dense network model. More specifically, we consider a wireless multihop network with a total of n nodes *i.i.d.* on a unit square $A=\left[-\frac{1}{2},\frac{1}{2}\right]^2$ following a general density function $f(\boldsymbol{x})$ where

$$f(\boldsymbol{x})>0 \qquad \forall \boldsymbol{x}\in A$$
$$\int_A f(\boldsymbol{x})\,d\boldsymbol{x}=1$$

Further, a pair of nodes are directly connected if and only if (iff) their Euclidean distance is smaller than or equal to $r(n)$, known as the transmission range, and the link capacity of each link is W bits/s. Here the value of $r(n)$ is assumed to be known and such that it ensures the resulting network is connected, which is a prerequisite for studying the capacity of the network. For simplicity, we may drop the dependence on n for notational brevity and use r for $r(n)$.

We consider a scenario where each node chooses randomly and independently another node in the network as its destination. Therefore there are a total of n source-destination pairs in the network. Further, a saturated traffic scenario is considered where each node always has a packet to transmit when transmission opportunity becomes available. Each node transmits following a CSMA protocol. That is, before transmitting, a node first senses the channel and can only transmit if there is no other active transmitter within $(1+\triangle)\,r(n)$, where $\triangle$ is a positive constant and $(1+\triangle)\,r(n)$ is commonly known as the sensing range. Thus two simultaneously active transmitters must be separated by a distance of at least $(1+\triangle)\,r(n)$. Noting that in the widely used protocol model or SINR model [1], a distance can also be identified such that two simultaneously active transmitters must be separated by at least that distance. Therefore the communication model used in this paper can be readily extended to incorporate other models too. Given the transmission power, the path loss model and the carrier-sensing threshold, the sensing range can be easily computed. We refer readers to [9] and our previous work [20] for more details of CSMA protocols. Further, a random backoff mechanism, which is commonly adopted in CSMA protocols, is used to resolve channel contention when multiple nodes contend for transmission. Therefore we consider an ideal scenario where there is no packet loss due to collision.

Denote the above network by $\mathcal{G}(n,r,A)$. In this work, we are interested in finding the capacity of $\mathcal{G}(n,r,A)$.

Particularly, we study the capacity of $\mathcal{G}(n,r,A)$ by investigating the so-called per-node throughput. Let Φ be the set of all spatial and temporal routing algorithms. Let $\lambda_i^{\chi}(n)$ be the *long-term* average throughput obtained by the i^{th} source-destination pair when using routing algorithm $\chi \in \Phi$. The per-node throughput of $\mathcal{G}(n, r, A)$ when using routing algorithm $\chi \in \Phi$, denoted by $\lambda^{\chi}(n)$, is given by

$$\lambda^{\chi}(n) = \min_{i \in \Gamma} \lambda_i^{\chi}(n)$$

where Γ is the set of indices of all source-destination pairs. The per-node throughput of $\mathcal{G}(n, r, A)$, denoted by $\lambda(n)$, is given by

$$\lambda(n) = \max_{\chi \in \Phi} \left(\lambda^{\chi}(n)\right) = \max_{\chi \in \Phi} \left(\min_{i \in \Gamma} \lambda_i^{\chi}(n)\right)$$

That is, there exists an optimum routing algorithm and a sufficiently large time interval τ such that every τ time interval, each and every source can transmit at least $\lambda(n)\tau$ bits to its destination simultaneously with all other source-destination pairs in the network. This definition is consistent with that used in [1].

III. Analysis of Capacity Upper Bound

The set of concurrent transmitters using the CSMA protocol is commonly modeled by a marked point process, known as Matern process or the hardcore process [15], [21]. Particularly, let $\boldsymbol{x}_i$ be the location of node i. Assume that time is divided into time slots of equal length. At the beginning of a time slot, a random number uniformly distributed within $[0, 1]$ is assigned to each node. Denote the number assigned to node i by p_i and it is assumed that p_i and p_j are independent where $i \neq j$. A node i is allowed to transmit if [21]

$$p_i < \min_{j \in \{k: \|\boldsymbol{x}_k - \boldsymbol{x}_i\| \leq (1+\triangle)r\}/\{i\}} p_j \tag{1}$$

That is, its p_i is the smallest among all nodes within its sensing range. In the next time slot, the above process repeats.

We will use a disk method to determine the capacity upper bound.

Let $D(R)$ be a disk centered at the origin and with a radius R where $0 < R \leq \frac{1}{2}$, as illustrated in Fig. 1. For an arbitrarily chosen node, the probability that the node falls within $D(R)$ is given by

$$\eta(R) = \int_{D(R)} f(\boldsymbol{x})\, d\boldsymbol{x} \tag{2}$$

and the probability that the node falls outside $D(R)$ is given by $1 - \eta(R)$. It follows that the expected number of nodes within $D(R)$ is $\eta(R)n$. Among these nodes within $D(R)$, with probability $1 - \eta(R)$, its destination is located outside $D(R)$. Therefore the expected fraction of source-destination pairs with the source and the destination located in different side of $D(R)$ is given by $2\eta(R)(1 - \eta(R))$.

Denote by $m(R)$ the expected number of active links crossing the boundary of $D(R)$, the following inequality must hold: $2\eta(R)(1-\eta(R))n\lambda(n) \leq m(R)W$ for any value of R so long as $D(R) \subset A$. As an easy consequence of the above inequality

$$\lambda(n) \leq \min_{0<R\leq 0.5} \frac{m(R)W}{2\eta(R)(1-\eta(R))n} \tag{3}$$

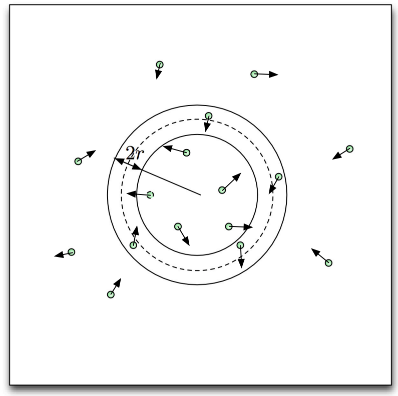

Figure 1. An illustration of the disk method of computing the capacity upper bound. The arrow represents the direction of packet forwarding.

In the following, we will try to find the value of $m(R)$.

Consider Fig. 1 where the boundary of $D(R)$ is shown as the dotted circle. Let $D(R-r, R+r)$ be an annulus centered at the origin and with an inner radius of $R-r$ and an outer radius of $R+r$. Let $L(R-r, R+r)$ be the random number of simultaneously active transmitters within $D(R-r, R+r)$ in a randomly chosen time slot. It follows that

$$m(R) \leq E(L(R-r, R+r)) \tag{4}$$

where the expectation $E()$ is taken with regards to time because the number of simultaneously active transmitters may be time-varying.

Next we shall find $E(L(R-r, R+r))$. Let dA be a very small (differential) area and $dA \subset D(R-r, R+r)$. Let $\boldsymbol{x}$ be the center of dA. Denote by η_{dA} the event that there is at least one node in dA. It can be shown that

$$\Pr(\eta_{dA}) = 1 - (1 - f(dA))^n = nf(dA) + o_{dA}(f(dA))$$

For convenience, we use $f(dA)$ for $\int_{dA} f(\boldsymbol{x})\, d\boldsymbol{x}$. When $dA \to 0$, $f(dA) \to f(\boldsymbol{x})\, dA$.

Let $D(\boldsymbol{x}, (1+\triangle)r)$ be a disk centered at the center of dA and with a radius $(1+\triangle)r$. The pmf (probability mass function) of the random number of nodes falling into $D(dA, (1+\triangle)r)$, denoted by $N(dA, (1+\triangle)r)$, is given by:

$$\begin{aligned}&\Pr(N(dA, (1+\triangle)r) = k)\\ =&\binom{n}{k}\left[f(D(\boldsymbol{x}, (1+\triangle)r))\right]^k \left[1 - f(D(\boldsymbol{x}, (1+\triangle)r))\right]^{n-k}\end{aligned}$$

Using the above equation and the expression for $\Pr(\eta_{dA})$ obtained earlier, the joint distribution $\Pr(N(dA, (1+\triangle)r) = k, \eta_{dA})$ is then given by:

$$\begin{aligned}&\Pr(N(dA, (1+\triangle)r) = k, \eta_{dA})\\ =&\Pr(\eta_{dA} | N(dA, (1+\triangle)r) = k) \Pr(N(dA, (1+\triangle)r) = k)\\ =&[1 - (1 - \frac{f(dA)}{f(D(\boldsymbol{x}, (1+\triangle)r))})^k] \Pr(N(dA, (1+\triangle)r) = k)\\ =&n\binom{n-1}{k-1}\left[f(D(\boldsymbol{x}, (1+\triangle)r))\right]^{k-1}\\ &\times\left[1 - f(D(\boldsymbol{x}, (1+\triangle)r))\right]^{n-k} f(dA)\end{aligned}$$

$$+o_{dA}\left(\frac{f(dA)}{f(D(\boldsymbol{x},(1+\triangle)r))}\right)$$

Conditioned on $N(dA,(1+\triangle)r)=k,\eta_{dA}$, denoting by ξ_{dA} the event that there is exactly one node in dA *and* that node is an active transmitter[2], it can be obtained that

$$\begin{aligned}&\Pr\left(\xi_{dA}|N(dA,(1+\triangle)r)=k,\eta_{dA}\right)\\=&\int_0^1(1-x)^{k-1}dx=\frac{1}{k}\end{aligned}$$

The above equation implies that the node in dA will have equal opportunity to transmit compared with other nodes in its sensing range.

From the above analysis, it follows that

$$\begin{aligned}&\Pr(\xi_{dA})\\=&\Pr(\xi_{dA},\eta_{dA})\\=&\sum_{k=1}^{n}\Pr(\xi_{dA},N(dA,(1+\triangle)r)=k,\eta_{dA})\\=&\sum_{k=1}^{n}[\Pr(\xi_{dA}|N(dA,(1+\triangle)r)=k,\eta_{dA})\\&\times\Pr(N(dA,(1+\triangle)r)=k,\eta_{dA})]\\=&\sum_{k=1}^{n}\frac{1}{k}\{[f(D(\boldsymbol{x},(1+\triangle)r))]^k[1-f(D(\boldsymbol{x},(1+\triangle)r))]^{n-k}\\&\times\binom{n}{k}k\frac{f(dA)}{f(D(\boldsymbol{x},(1+\triangle)r))}+o_{dA}\left(\frac{f(dA)}{f(D(\boldsymbol{x},(1+\triangle)r))}\right)\}\\=&\frac{1}{f(D(\boldsymbol{x},(1+\triangle)r))}\sum_{k=1}^{n}\{\binom{n}{k}[f(D(\boldsymbol{x},(1+\triangle)r))]^k\\&\times[1-f(D(\boldsymbol{x},(1+\triangle)r))]^{n-k}\}f(dA)+o_{dA}(f(dA))\\=&\frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))}f(dA)+o_{dA}(f(dA))\end{aligned}\tag{5}$$

It follows from the above equation that

$$\begin{aligned}&E(L(R-r,R+r))\\=&\int_{L(R-r,R+r)}\frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))}f(dA)\\=&\int_{L(R-r,R+r)}\frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))}f(\boldsymbol{x})\,dA\end{aligned}\tag{6}$$

Summarizing the four equations (2), (3), (4) and (6), a main result on the capacity upper bound of networks with generally distributed nodes is obtained:

$$\begin{aligned}&\lambda(n)\\\le&\min_{0<R\le0.5}\frac{W\int_{L(R-r,R+r)}\frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))}f(\boldsymbol{x})\,dA}{2n\int_{D(R)}f(\boldsymbol{x})\,d\boldsymbol{x}\left(1-\int_{D(R)}f(\boldsymbol{x})\,d\boldsymbol{x}\right)}\end{aligned}\tag{7}$$

Remark 1. In the analysis, we use a disk, i.e. $D(R)$, mainly for convenience. In (7), $D(R)$ can be replaced by a connected area of any shape entirely contained in A and the inequality on the capacity upper bound will remain valid.

[2]Following some simple argument, it can be shown that when $dA\to0$, the probability that there is more than one node in dA is negligible.

IV. Discussion on the Capacity Upper Bound

The capacity upper bound obtained in the last section, particularly in (7), is valid for both finite n and asymptotically infinite n, and for both uniformly and Poissonly distributed nodes and nodes distributed following a more general distribution. The analytical form of the capacity upper bound in (7) is however complicated and is difficult to extract key information on the impact of major capacity-impacting factors. Therefore, in this section, by imposing some mild conditions on the transmission range $r(n)$, we simplify the upper bound in (7) in a bid to extract the major factors that determine the capacity and their roles.

Particularly, we assume that in the asymptotic regime when $n\to\infty$, $r(n)$ satisfies the two conditions that

$$r(n)\to0\tag{8}$$

$$r(n)=\omega_n\left(\frac{1}{\sqrt{n}}\right)\tag{9}$$

Recall that in this paper, a dense network model is used. It is a natural outcome in the dense network model that, as $n\to\infty$, a smaller and smaller transmission range, i.e. $r(n)\to0$, is used to improve the spatial frequency reuse and hence the capacity; otherwise the entire network may end up being able to have a small and constant number (i.e. does not increase with n) of simultaneous transmissions only, despite a large number of nodes competing for channel access.

The condition in (9) that $r(n)=\omega_n\left(\frac{1}{\sqrt{n}}\right)$ is a necessary condition for the network $\mathcal{G}(n,r,A)$ to be a.a.s. connected. We illustrate this by evaluating the number of isolated nodes in $\mathcal{G}(n,r,A)$. A node is isolated if there is no other node in its transmission range.

Consider a differential area $dA\subset A$, the probability that there is exactly one node in the area is given by $nf(dA)(1-f(dA))^{n-1}$ and the probability that there is more than one node in the area is negligible. Denote by η_{dA} the event that there is exactly one node in dA and denote by ζ_{dA} the event that there is a node in dA *and* the node is isolated. Let $\boldsymbol{x}$ be the center of dA. Without loss of generality, we assume that when there is a node in dA, that node is located at $\boldsymbol{x}$ (Because we are considering a differential area, the actual position of the node in the area does not matter.) It can be shown that

$$\begin{aligned}&\Pr(\zeta_{dA})\\=&\Pr(\zeta_{dA}|\eta_{dA})\Pr(\eta_{dA})\\=&[1-f(D(\boldsymbol{x},r)\setminus dA)]^{n-1}nf(dA)(1-f(dA))^{n-1}\end{aligned}$$

Denoted by ζ the number of isolated nodes in the network. It can be shown:

$$\begin{aligned}&E(\zeta)\\=&\int_A[1-f(D(\boldsymbol{x},r)\setminus dA)]^{n-1}nf(dA)\\=&\int_A[1-f(D(\boldsymbol{x},r))]^{n-1}nf(\boldsymbol{x})\,dA\end{aligned}$$

$$
\begin{aligned}
&= \int_A e^{(n-1)\log[1-f(D(\boldsymbol{x},r))]} nf(\boldsymbol{x})\,dA \\
&\sim_n \int_A e^{-(n-1)f(\boldsymbol{x})\pi r^2} nf(\boldsymbol{x})\,dA
\end{aligned}
$$

where in the last step, the equations that $f(D(\boldsymbol{x},r)) \sim_r f(\boldsymbol{x})\pi r^2$ as $r \to 0$ and that $\log(1-x) \sim_x -x$ as $x \to 0$ are used. Let $c_2 = \max_{\boldsymbol{x}\in A} f(\boldsymbol{x})$, it follows from the above equation that

$$
\begin{aligned}
&\lim_{n\to\infty} E(\zeta) \\
\geq\ & \lim_{n\to\infty} \int_A e^{-(n-1)c_2\pi r^2} nf(\boldsymbol{x})\,dA \\
=\ & \lim_{n\to\infty} ne^{-(n-1)c_2\pi r^2} \qquad (10)
\end{aligned}
$$

Equation (10) clearly shows that if the condition in (9) is *not* fulfilled, the expected number of isolated nodes will increase towards infinity as $n \to \infty$. In fact, as suggested in (10), a large fraction of the nodes will be isolated and the network will be quite fragmented. Therefore the condition in (9) is required for the network to be a.a.s. connected as $n \to \infty$.

After having established that the conditions in (8) and (9) are reasonable conditions to be imposed on $r(n)$, we now examine the capacity upper bound established in (7) as $n \to \infty$. Particularly under the two conditions, it can be shown that

$$
f(D(\boldsymbol{x},(1+\triangle)r)) \sim_n f(\boldsymbol{x})\pi(1+\triangle)^2 r^2
$$

and

$$
\begin{aligned}
&\left[1 - f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]^n \\
=\ & e^{n\log\left[1-f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]} \\
\sim_n\ & e^{-nf(\boldsymbol{x})\pi(1+\triangle)^2 r^2}
\end{aligned}
$$

Therefore, $\left[1 - f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]^n \to 0$ as $n \to \infty$ as a consequence of (9). Using the above two results and (7), it follows that:

$$
\begin{aligned}
&\lambda(n) \\
\leq\ & \min_{0<R\leq 0.5} \frac{W\int_{L(R-r,R+r)} \frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))} f(\boldsymbol{x})\,dA}{2n\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left(1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right)} \\
\leq\ & \min_{0<R\leq 0.5} \frac{\frac{4R}{(1+\triangle)^2 r} W}{2n\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \\
=\ & \min_{0<R\leq 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} W\frac{1}{n}\frac{1}{r} \quad (11)
\end{aligned}
$$

where the second step results because of the following derivation:

$$
\begin{aligned}
& \int_{L(R-r,R+r)} \frac{1-[1-f(D(\boldsymbol{x},(1+\triangle)r))]^n}{f(D(\boldsymbol{x},(1+\triangle)r))} f(\boldsymbol{x})\,dA \\
\sim_n\ & \int_{L(R-r,R+r)} \frac{1-\left[1-f(\boldsymbol{x})\pi(1+\triangle)^2 r^2\right]^n}{f(\boldsymbol{x})\pi(1+\triangle)^2 r^2} f(\boldsymbol{x})\,dA \\
\sim_n\ & \int_{L(R-r,R+r)} \frac{1}{\pi(1+\triangle)^2 r^2}\,dA \\
=\ & \frac{4R}{(1+\triangle)^2 r}
\end{aligned}
$$

Let

$$
\beta_f \triangleq \min_{0<R\leq 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \qquad (12)
$$

where the subscript f emphasizes the dependence of β on the node distribution function f, (11) can be written in a more neatly form as

$$
\lambda(n) \leq \beta_f W \times \frac{1}{n} \times \frac{1}{r} \qquad (13)
$$

In (13), parameter β_f captures the impact of node distribution and is also entirely determined by the node distribution. Parameter W represents the impact of link capacity. Parameter $\frac{1}{n}$ represents the impact of the number of source-destination paris sharing the network capacity. Finally parameter $\frac{1}{r}$ represents the impact of the transmission range and shows that the network capacity (upper bound) is inversely proportional to the transmission range. Equation (13) shows that the network capacity (upper bound) can be expressed as the product of the above four terms. Particularly, *other things being equal*, different node distribution has an impact of up to a constant factor (i.e. not varying with n) on the network capacity only.

A. Discussion on the tightness of the upper bound

In this subsection, we check the tightness of the capacity upper bound by first applying the upper bound to a special case, i.e. networks with uniformly distributed nodes, then comparing the obtained upper bound for networks with uniformly distributed node with known results in the area.

For networks with uniformly distributed nodes, $f(\boldsymbol{x}) = 1$. It follows from (12) that

$$
\begin{aligned}
\beta_f &= \min_{0<R\leq 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \\
&= \min_{0<R\leq 0.5} \frac{\frac{2R}{(1+\triangle)^2}}{\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\left[1-\int_{D(R)} f(\boldsymbol{x})\,d\boldsymbol{x}\right]} \\
&= \min_{0<R\leq 0.5} \frac{2}{\pi(1+\triangle)^2} \times \frac{1}{R(1-\pi R^2)} \\
&= \frac{4}{3\pi(1+\triangle)^2\sqrt{3\pi}}
\end{aligned}
$$

Combining the above equation with (13), an upper bound on the per-node throughput of networks with uniformly distributed nodes results:

$$
\lambda(n) \leq \frac{4}{3\pi(1+\triangle)^2\sqrt{3\pi}} W \times \frac{1}{n} \times \frac{1}{r} \qquad (14)
$$

That is, the capacity upper bound is inversely proportional to the transmission range. It is well known that the critical transmission range required for a network with a total of n nodes uniformly distributed in a unit area to be a.a.s. connected is [22], [23]

$$
r^c(n) = \sqrt{\frac{\log n + c(n)}{\pi n}} \qquad (15)
$$

where $c(n) = o_n(\log n)$ and $c(n) \to \infty$ as $n \to \infty$.

As an easy consequence of (14) and (15),

$$\lambda(n) \leq \frac{4}{3\pi(1+\triangle)^2\sqrt{3}} W \times \frac{1}{\sqrt{n(\log n + c(n))}}$$

It is well known that the per-node throughput of networks with uniformly distributed node is $\Theta_n\left(\frac{W}{\sqrt{n\log n}}\right)$ [1]. Therefore the capacity upper bound obtained in (13) is tight in the sense that when applied to the special case of networks with uniformly distributed nodes, the capacity upper bound is in the same order of the capacity of networks with uniformly distributed nodes.

V. Conclusion

In this paper, we studied the capacity of networks with generally distributed nodes. More specifically, we considered networks with a total of n nodes i.i.d. on a unit square following a general distribution function. Further, a pair of nodes are directly connected following a unit disk model with a transmission range $r(n)$ and two simultaneous active transmitters have to be separated by at least $(1+\triangle)r(n)$ due to the use of carrier sensing. The capacity of each link is W bits/s. A capacity upper bound of the above network is obtained by analyzing the number of links crossing a simple closed curve. The capacity upper bound is valid for both finite networks and asymptotic infinite networks.

To simplify the analytical expression and to gain insight, we imposed some mild conditions on the transmission range $r(n)$ which were shown to be reasonable conditions required for the transmission range to meet. Under these additional conditions, the (asymptotic) capacity upper bound is shown to be the product of four terms: the first term β_f determined by the spatial node distribution only, the second term W representing the link capacity, the third term $\frac{1}{n}$ where n represents the number of source-destination pairs sharing the network capacity, and the fourth term $\frac{1}{r(n)}$ representing the combined impact of spatial frequency reuse and the number of relay hops required for end-to-end information delivery. It implies that the impact of the spatial node distribution on the network capacity can be captured by a single parameter and the spatial node distribution only affect the network capacity by up to a constant factor. The tightness of the capacity upper bound was validated by first applying the upper bound to the special case of networks with uniformly distributed nodes and then comparing the obtained upper bound with known results in the literature. The capacity upper bound was shown to be tight in the sense that for the special case of networks with uniformly distributed nodes, the capacity upper bound is in the same order as known results in the literature.

We expect that not only the (asymptotic) capacity upper bound but also the (asymptotic) capacity of networks with generally distributed nodes can be expressed in the product form of the four factors, representing respectively the impact of spatial node distribution, the link capacity, the number of source-destination pairs and the combined impact of spatial frequency reuse and the number of relays required for end-to-end information delivery which is manifested through the single parameter of transmission range. It is part of our future work plan to validate the conjecture.

References

[1] P. Gupta and P. Kumar, "The capacity of wireless networks," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 388–404, 2000.

[2] M. Franceschetti, O. Dousse, D. N. C. Tse, and P. Thiran, "Closing the gap in the capacity of wireless networks via percolation theory," *IEEE Transactions on Information Theory*, vol. 53, no. 3, pp. 1009–1018, 2007.

[3] O. Dousse, P. Mannersalo, and O. Thiran, "Latency of wireless sensor networks with uncoordinated power savings mechanisms," in *Proceedings of the 5th ACM international symposium on Mobile ad hoc networking and computing (MOBIHOC 2004)*, pp. 109–120.

[4] M. Grossglauser and D. N. C. Tse, "Mobility increases the capacity of ad hoc wireless networks," *IEEE/ACM Transactions on Networking*, vol. 10, no. 4, pp. 477–486, 2002.

[5] A. E. Gamal, J. Mammen, B. Prabhakar, and D. Shah, "Throughput-delay trade-off in wireless networks," in *IEEE INFOCOM 2004*, vol. 1, pp. 464–475.

[6] A. Zemlianov and V. Gustavo de, "Capacity of ad hoc wireless networks with infrastructure support," *IEEE Journal on Selected Areas in Communications*, vol. 23, no. 3, pp. 657–667, 2005.

[7] S. Chen, Y. Wang, X.-Y. Li, and X. Shi, "Order-optimal data collection in wireless sensor networks: Delay and capacity," in *IEEE SECON 2009*, pp. 1–9.

[8] X.-Y. Li, "Multicast capacity of wireless ad hoc networks," *IEEE/ACM Transactions on Networking*, vol. 17, no. 3, pp. 950–961, 2009.

[9] C.-K. Chau, M. Chen, and S. C. Liew, "Capacity of large-scale csma wireless networks," *IEEE/ACM Transactions on Networking*, vol. 19, no. 3, pp. 893–906, 2011.

[10] R. Meester and R. Roy, *Continuum Percolation*, ser. Cambridge Tracts in Mathematics. Cambridge University Press, 1996.

[11] P. Jacquet, S. Malik, B. Mans, and A. Silva, "On the throughput-delay trade-off in georouting networks," in *IEEE INFOCOM 2012*, pp. 765–773.

[12] Z. Kong and E. M. Yeh, "Connectivity and latency in large-scale wireless networks with unreliable links," in *IEEE INFOCOM 2008*, pp. 11–15.

[13] P. Li, C. Zhang, and Y. Fang, "Capacity and delay of hybrid wireless broadband access networks," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 2, pp. 117–125, 2009.

[14] M. J. Neely and E. Modiano, "Capacity and delay tradeoffs for ad hoc mobile networks," *IEEE Transactions on Information Theory*, vol. 51, no. 6, pp. 1917–1937, 2005.

[15] M. Haenggi, J. G. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, "Stochastic geometry and random graphs for the analysis and design of wireless networks," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 7, pp. 1029–1046, 2009.

[16] G. Alfano, M. Garetto, and E. Leonardi, "Capacity scaling of wireless networks with inhomogeneous node density: upper bounds," *IEEE Journal on Selected Areas in Communications*, vol. 27, no. 7, pp. 1147–1157, 2009.

[17] G. Alfano, M. Garetto, E. Leonardi, and V. Martina, "Capacity scaling of wireless networks with inhomogeneous node density: Lower bounds," *IEEE/ACM Transactions on Networking*, vol. 18, no. 5, pp. 1624–1636, 2010.

[18] M. Garetto, A. Nordio, C. Chiasserini, and E. Leonardi, "Information-theoretic capacity of clustered random networks," *IEEE Transactions on Information Theory*, vol. 57, no. 11, pp. 7578–7596, 2011.

[19] M. Franceschetti and R. Meester, *Random Networks for Communication*. Cambridge University Press, 2007.

[20] T. Yang, G. Mao, and W. Zhang, "Connectivity of large-scale csma networks," *IEEE Transactions on Wireless Communications*, vol. 11, no. 9, pp. 2266 – 2275, 2012.

[21] F. Baccelli and B. Blaszczyszyn, *Stochastic Geometry and Wireless Networks, Part I: Theory*, ser. Foundations and Trends(r) in Networking. Now Publishers Inc, 2009.

[22] P. Gupta and P. R. Kumar, *Critical Power for Asymptotic Connectivity in Wireless Networks*. Boston, MA: Birkhauser, 1998, pp. 547–566.

[23] G. Mao and B. Anderson, "Connectivity of large wireless networks under a general connection model," *IEEE Transactions on Information Theory*, vol. 59, no. 3, pp. 1761– 1772, 2013.