On the Probability of K-hop Connection in Wireless Sensor Networks

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***Abstract*— Considering a wireless sensor network whose sensors are randomly and independently distributed in a bounded area according to a homogeneous Poisson process with intensity ρ and each sensor has a uniform transmission radius of r_0, an important problem is to obtain the probability that two random sensors separated by a known distance x are k hop neighbors for some positive integer k. In this letter, we give a recursive analytical equation for computing this probability. The analytical solution is validated by simulations.**

***Index Terms*— Probability, k-hop connection, wireless sensor networks.**

I. Introduction

CONSIDER a wireless sensor network (WSN) whose sensors are randomly and independently distributed in a bounded region in $\Re^2$, according to a homogeneous Poisson process. Each sensor has a uniform transmission radius of r_0. This sensor network can be modeled by a unit disk graph with a vertex in the graph uniquely representing every sensor and an edge in the graph representing every sensor pair for which the two sensors can directly communicate with each other. Any two sensors can directly communicate with each other if and only if their Euclidean distance is smaller than the given threshold r_0. The resulting graph $G = (V, E)$, where V is the vertex set and E is the edge set, is called the *underlying graph* of the network. Many interesting aspects of a WSN can be studied using its underlying graph.

In this letter we investigate the conditional probability $Pr(k|x)$ that any two sensors separated by a known distance x are k-hop neighbors for some positive integer k. Two sensors being k-hop neighbors means that the length of the shortest path between the two sensors, measured in the number of hops, is k. A recursive analytical equation embodying an approximation is given for computing this probability. The results are important because they enable solution of a number of problems, such as estimation of the energy consumption for data transmission between two sensors and the associated transmission delay. Further, given the probability density $p(x)$ for the distance x between any two sensors [1], our result can be used to compute the unconditional probability $Pr(k)$, which is useful for studying problems such as estimating the overall energy consumption and the lifetime of a WSN, and WSN routing problems. Again, once $Pr(k|x)$ has been obtained, $Pr(x|k)$ can be readily found using Bayes' formula. Knowledge of $Pr(x|k)$ may be used to improve the performance of WSN localization algorithms, e.g., connectivity-based localization algorithms [2].

In [3], Bettstetter and Eberspaecher investigated the probability that two random sensors are k-hop neighbors for $k = 1$ and $k = 2$, where sensors are uniformly distributed in a rectangular area. For $k > 2$, only simulation results were presented. In [4], Miller considered sensors distributed following a two dimensional Gaussian distribution, and derived an approximation for the probability that two random sensors are 2-hop neighbors. In [5], Chandler analyzed the probability two random packet radio stations separated by a known distance can communicate in k or less hops where stations are uniformly distributed over flat earth.

In this letter, we provide a recursive equation for computing the probability that two random sensors are exact k-hop neighbors for some positive integer k. The technique used in deriving the recursive equation is the same as that used in [5]. A contribution of this letter is we point out the reliance of the analysis on the *independence assumption*, which is defined later, and the performance impact of the independence assumption. Furthermore, we show that the probability that two random sensors are k-hop neighbors is only determined by two parameters, i.e., the normalized distance and the average vertex degree. We also provide simulation results as well as discrepancy analysis between the simulation results and the analytical results.

In Section II we derive the probability that any two sensors separated by a known distance x are two-hop neighbors. In Section III we develop a recursive analytical equation for the probability that any two sensors separated by a known distance x are k-hop neighbors for $k > 2$. Section IV presents simulation results. Finally, Section V concludes this letter and discusses future work.

II. Probability That Any Two Sensors are Two-Hop Neighbors

In what follows, the conditional probability $Pr(k|x)$ that two random sensors separated by a known distance x are k-hop neighbors is denoted by $\Phi_k(x)$. The disk of radius l centered at s is denoted by $D(s, l)$.

Obviously, $\Phi_1(x) = 1$ when $x \leq r_0$ and $\Phi_1(x) = 0$ when $x > r_0$. For $k = 2$, it means that the source s and the destination d have no direct link between each other but can communicate through at least one intermediate sensor. Therefore at least one sensor must lie in the intersectional area $D(s, r_0) \cap D(d, r_0)$, i.e., the shaded area $\mathbf{A}$ in Fig. 1. Ignoring the boundary effect, the probability $\Phi_2(x)$ can be found as the probability that there is at least one sensor located in $\mathbf{A}$.

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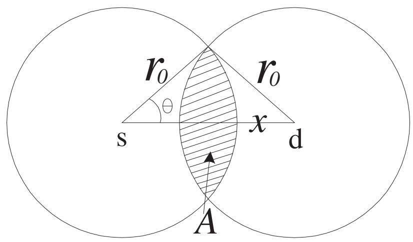

Fig. 1. An illustration of two-hop neighbors. Sensors s and d are two random sensors separated by a distance x ($r_0 < x \leq 2r_0$).

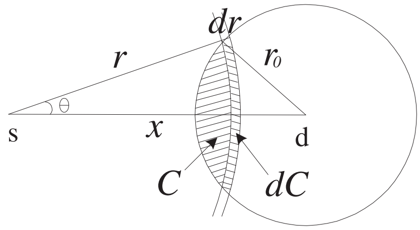

Fig. 2. An illustration of k-hop ($k > 1$) neighbors. Sensors s and d are two random sensors separated by a known distance x ($x > r_0$).

Since sensors are Poissonly distributed, the probability that there is no sensor located in **A** is $exp(-\rho A)$, where A is the size of the area **A**, given by

$$A = 2r_0^2 \arcsin(\sqrt{1 - \frac{x^2}{4r_0^2}}) - xr_0\sqrt{1 - \frac{x^2}{4r_0^2}}, \quad r_0 < x \leq 2r_0. \tag{1}$$

Hence, the probability $\Phi_2(x)$ can be readily obtained:

$$\Phi_2(x) = 1 - Pr\{\texttt{no sensor in } \mathbf{A}\} = 1 - e^{-\rho A}. \tag{2}$$

When $x \leq r_0$, the two sensors can connect directly with each other, so $\Phi_2(x) = 0$; when $x > 2r_0$, we have $A = 0$, so that, $\Phi_2(x) = 0$. Therefore,

$$\Phi_2(x) = \begin{cases} 1 - e^{-\rho A}, & r_0 < x \leq 2r_0; \\ 0, & \texttt{otherwise}. \end{cases} \tag{3}$$

III. Probability That Any Two Sensors are k-hop Neighbors

In this section, we shall evaluate the probability $\Phi_k(x)$ for $k > 2$, continuing to ignore the boundary effect. Consider two random sensors s and d separated by distance x, as shown in Fig. 2. Sensor d is a k-hop neighbor of s iff sensor d is not a m-hop neighbor of s for any $m < k$ *and* there is at least one sensor within $D(d, r_0)$ which is a $k-1$ hop neighbor of s.

Let us first consider the probability that there is at least one sensor within $D(d, r_0)$ which is a $k-1$ hop neighbor of s. An approximation, termed the *independence assumption* (and verified as reasonable by simulation evidence below), has to be used in order to obtain an analytical solution, i.e., the event that one arbitrary sensor located within $D(d, r_0)$ is a $k-1$ hop neighbor of s is independent of the event that another arbitrary sensor located within $D(d, r_0)$ is a $k-1$ hop neighbor of s. Consider the area C in Fig. 2, i.e., $C = D(s, r) \cap D(d, r_0)$, a differential increment dr on r gives a differential area dC and the size of the differential area dC is $dC = 2r\theta dr$, where θ is given by,

$$\theta = \arccos \frac{x^2 + r^2 - r_0^2}{2xr}. \tag{4}$$

Since dr is a very small value, the probability that there exists more than one sensor within dC can be ignored and the probability that there exists a single sensor in the differential area dC is given by $2\rho r\theta dr$. Given $\Phi_{k-1}(r)$, the probability that there is a sensor within dC which is also a $k-1$ hop neighbors of s is given by $2\Phi_{k-1}(r)\rho r\theta dr$.

Let $f(C)$ denote the probability that there is *no* sensor in C which is a $k-1$ hop neighbor of s. Then

$$f(C + dC) = f(C)(1 - 2\Phi_{k-1}(r)\rho r\theta dr). \tag{5}$$

Eq. 5 readily leads to the conclusion that

$$df(C) = -2\Phi_{k-1}(r)\rho r\theta f(C)dr. \tag{6}$$

Therefore the probability that there is *no* sensor within $D(d, r_0)$ which is a $k-1$ hop neighbor of s at distance x is given by

$$g(x) = exp(\int_{x-r_0}^{x+r_0} -2\Phi_{k-1}(r)\rho r\theta dr). \tag{7}$$

The probability that a sensor d at a distance x to s is not a m-hop neighbor of s for any $m < k$ is given by:

$$1 - \sum_{i=1}^{k-1} \Phi_i(x). \tag{8}$$

Therefore the probability that sensor d is a k-hop neighbor of s is given recursively as:

$$\Phi_k(x) = (1-\sum_{i=1}^{k-1} \Phi_i(x))(1-\exp(-\int_{x-r_0}^{x+r_0} 2\Phi_{k-1}(r)\rho r\theta dr)). \tag{9}$$

For $k = 2$, when $x \leq r_0$ or $x > 2r_0$, it can be readily shown that $\Phi_2(x) = 0$; when $r_0 < x \leq 2r_0$, it can be shown that

$$-\int_{x-r_0}^{x+r_0} 2\Phi_1(r)\rho r\theta dr - \rho \int_{x-r_0}^{r_0} 2r\theta dr = -\rho A, \tag{10}$$

where A is given in Eq. 1, and $1 - \Phi_1(x) = 1$, the expression for $\Phi_2(x)$ agrees with that in Eq. 3.

A. Discussion

Let $\alpha \doteq x/r_0$ and $\beta \doteq \pi r_0^2 \rho$. The parameter α is the normalized distance and the parameter β is the average vertex degree of a unit disk graph. It can be shown by *mathematical induction* that *under the independence assumption, $\Phi_k(x)$ is only determined by the normalized distance and the average vertex degree.*.

Proof: For $k = 1$, it is immediate that $\Phi_1(x)$ is a function of α only. Suppose $\Phi_n(x)$ can be expressed as a function of α and β for $n \leq k$, i.e., $\Phi_n(x) = \Upsilon_n(\alpha, \beta)$. Then when $n = k + 1$, the first term on the right side of Eq. 9 is

$$1 - \sum_{i=1}^{k} \Phi_i(x) = 1 - \sum_{i=1}^{k} \Upsilon_i(\alpha, \beta). \tag{11}$$

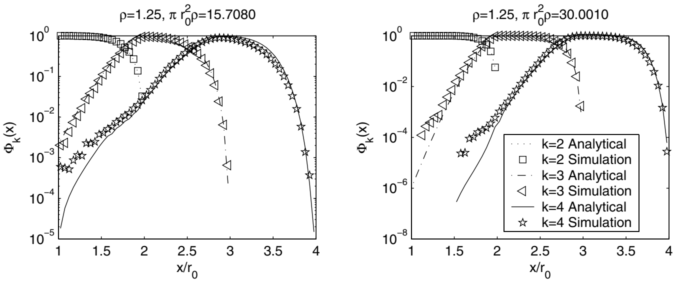

Fig. 3. Probability that two random sensors separated by a known distance x are k-hop neighbors for $k = 2, 3$ and 4.

Let $\mu = r/r_0$, the integral in the second term on the right side of Eq. 9 becomes

$$\begin{aligned} & \int_{x-r_0}^{x+r_0} 2\Upsilon_k(\frac{r}{r_0}, \beta)\rho r\theta dr \\ = & \int_{x/r_0-1}^{x/r_0+1} 2\Upsilon_k(\mu, \beta)\rho\mu r_0\theta r_0 d\mu \\ = & \frac{\beta}{\pi}\int_{\alpha-1}^{\alpha+1} 2\Upsilon_k(\mu, \beta) \arccos\frac{\alpha^2+\mu^2-1}{2\alpha\mu}\mu d\mu. \quad (12) \end{aligned}$$

From Eq. 9, 11 and 12, we have $\Phi_{k+1}(x) = \Upsilon_{k+1}(\alpha, \beta)$, hence, the hypothesis is also valid when $n = k + 1$.

IV. Simulation

In this section, we use simulations to establish the accuracy of the theoretical analysis in the presence of the boundary effect and the shortcomings of the independence assumption. In the simulation, sensors are distributed in a square of size $a \times a$, where $a = 20$, according to a homogenous Poisson process with node density ρ. We vary the average vertex degree (i.e., $\pi r_0^2 \rho$) while keeping the node density ρ fixed, each value of the average vertex degree represents a different scenario. Each scenario is repeated 100 times and the average result is shown. The analytical results are obtained through numerical integration using the adaptive quadrature algorithm [6, pp.27-41], which calculates more points only in regions of rapid functional variation and less points where the integrand is varying slowly, and hence obtains high accuracy of numerical results with given constraints on the computational complexity.

Fig. 3 displays $\Phi_k(x)$ for $k = 2, 3$ and 4. For $k = 2$, we can see that the simulation results and the analytical results agree very well, which indicates that Eq. 3 is an accurate expression of $\Phi_2(x)$. However, for $k = 3$ and 4, there are slight discrepancies between the analytical results and the corresponding simulation results, as shown in Fig. 3. The discrepancies are attributable to the *boundary effect* and the *independence assumption* we used in Section III. Note that the discrepancy for the small probabilities (e.g., $\Phi_k(x) \sim 10^{-5}$) is due to the accuracy of the numerical integration. Because the analytical result is given in a recursive form, it is also possible that the accuracy of the numerical integration decreases for large k. However, this was not found to be the major cause of the discrepancy in the simulation. The boundary effect is illustrated by Fig. 4. In Fig. 4, we keep $\pi r_0^2 \rho$ constant and vary the ratio r_0/a to calculate the mean absolute difference (MAD) between the analytical results and the simulation results. MAD is the average value of the absolute differences, i.e., $MAD = \frac{1}{N}\sum_{i=1}^{N} |Ana_i - Sim_i|$, where Ana_i and Sim_i are the i-th analytical result and its corresponding simulation result respectively, and N is the number of results selected to calculate MAD. The larger MAD is, the greater the discrepancy is.

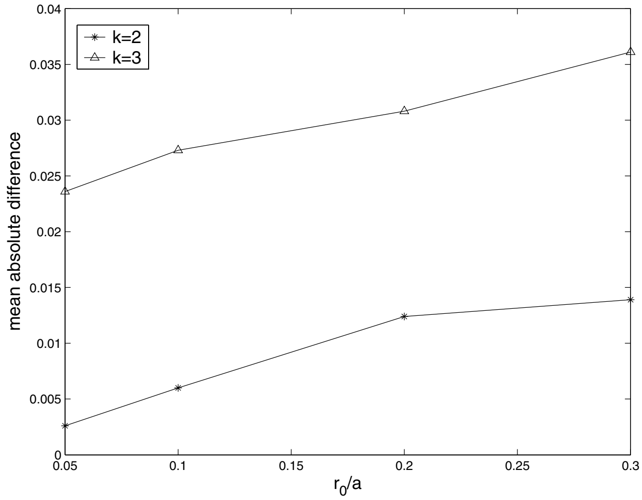

Fig. 4. MAD between the analytical results and the corresponding simulation results for different values of r_0/a; $\pi r_0^2 \rho$ is kept constant at 15.7080. When r_0/a increases, a larger percentage of sensors are located near the boundary and the impact of the boundary effect is more pronounced. Thus the discrepancy between the analytical and the simulation results increases.

V. Conclusion and Future Work

In this letter, we provided an analytical recursive equation for the probability that any two sensors separated by a known distance x are k-hop neighbors for any positive integer k. Simulations were conducted which validated the accuracy of the approximate analytical equation.

In the future, we shall consider sensors distributed following a uniform distribution and the impact of the boundary effect on the analytical equation.

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