Phase Transition Properties in K-connected Wireless Multi-hop Networks

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***Abstract*—Consider a wireless multi-hop network formed by distributing a total of n nodes randomly and uniformly in the unit cube $[0,1]^d$ ($d=1,2,3$) and connecting any two distinct nodes directly iff (if and only if) their Euclidean distance is not greater than a given threshold $r(n)$. We study the phase transition phenomenon of a k-connected ($k \in \mathbb{N}$) multi-hop network in this paper. We show that the phase transition of k-connectivity becomes sharper as n increases. We derive a generic analytical formula for the *phase transition width* for large n and for any fixed $k \in \mathbb{N}$ in d-dimensional space. The result in this paper is important for understanding phase transition behavior, and it provides valuable insight into the design and implementation of wireless multi-hop networks.**

I. Introduction

In a wireless multi-hop network, e.g., wireless sensor/ad hoc networks, most network functions rely on the underlying network being connected (or 1-connected) [1]. Some functions, such as unique localizability [2], robustness in routing [3], fault tolerance [4], etc, rely on the network being k-connected with $k>1$. A network is said to be connected iff for any pair of two nodes, there is at least one path between them. The network is k-connected iff the removal of any set of $(k-1)$ nodes will not disconnect the network. In other words, if any $(k-1)$ nodes fail in the network, the network still remains connected.

In a network with the nodes randomly and independently distributed in a bounded area, it has been shown that there exists a threshold in transmission range above which the network is k-connected with a high probability; and there exists another threshold in transmission range below which the network is k-connected with a low probability, i.e., the network is more likely not k-connected [1], [5]. The *difference* between the two thresholds defines the so-called *phase transition width*. We will give a more rigorous definition of the phase transition width shortly. Intuitively, the phase transition width gives an indication on how easy/difficult it is to transform a network that is not k-connected into a k-connected network. It has been shown that the phase transition width becomes sharper as the total number of nodes n increases [1], [5], [6], [7]. A good understanding of such a phase transition phenomenon is of practical significance for the design and implementation of wireless multi-hop networks.

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In this paper, we shall investigate analytically how quickly the phase transition of k-connectivity ($k \in \mathbb{N}$) happens in a wireless multi-hop network. We use a metric called the *phase transition width* to measure such rate. The phase transition width is also known as the *threshold width* in some papers [8]. Let $P_k(n, r(n))$ denote the probability that an instance of a randomly generated network with n nodes and a transmission range of $r(n)$ is k-connected. $P_k(n,r(n))$ is a strictly monotonically increasing function of $r(n)$ for $0 < P_k(n,r(n)) < 1$ in some finite interval of $r(n)$, and $P_k(n,r(n)) = 0$ or 1 outside the interval [5]. Let α denote a positive real number. Define

$$r_k(n,\alpha) := \inf(r>0: \ P_k(n,r) \geq \alpha), \ \alpha \in (0,1). \qquad (1)$$

The *phase transition width* over the probability interval $[\alpha, 1-\alpha]$ of k-connectivity is then defined as

$$\delta_k(n,\alpha) := r_k(n,1-\alpha) - r_k(n,\alpha), \ \alpha \in (0,\frac{1}{2}). \qquad (2)$$

Henceforth, unless otherwise indicated, the short term phase transition width will be used with α being simply understood.

Previous results derived on the phase transition width of k-connectivity are only valid for $k=1$ and $d=1,2$ [5], and more details of those results are given in section III. There is no *generic* result for all $k \in \mathbb{N}$ and $d \in \{1,2,3\}$. In this paper, we further develop the previous results and derive a *generic* analytical formula for the phase transition width of k-connectivity $\delta_k(n,\alpha)$ for large n and for any $k \in \mathbb{N}$ in d-dimensional space ($d=1,2,3$). The accuracy of the analytical results is verified using simulations. To the best of our knowledge, these results have never been presented before.

The rest of this paper is organized as follows. Section II describes the network model and some notations used in later analysis. Section III briefly reviews related work. Section IV presents the main results of the paper on $\delta_k(n,\alpha)$ for large n and for any $k \in \mathbb{N}$ in d-dimensional space ($d=1,2,3$). Section V presents the simulation results. Finally, Section VI concludes this paper and discusses possible future work.

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II. Preliminaries

A. Network model

In the past several years, *random geometric graphs* (RGG) have been widely used to model wireless multi-hop networks [1], [6], [9], [10], [11], [12]. Typically, a random geometric graph $G(n,r)$ is a graph in which n vertices are randomly and uniformly distributed in a unit cube $[0,1]^d$ $(d=1,2,3,...)$, and any two vertices u and v are directly connected iff $||u-v|| \le r$, where the norm $||\cdot||$ means the Euclidean norm. Due to the scaling property of the RGG, any realization $G(n, r(n))$ in a *unit* cube in $\Re^d$ coincides with another realization $G(n, \sqrt[d]{V}r(n))$ placed in a cube of volume V in $\Re^d$ [13].

Hence, throughout this paper, we model a network by a RGG $G(n,r(n))$ in a *unit* cube in $\Re^d$. In addition, we assume $n \gg 1$ and $1 \gg \pi_d r^d(n)$ so that the node distribution can be approximated by a homogeneous Poisson point process of intensity $\rho = n/1 = n$ [1], [13], [14], [15], [16]. The notation π_d is a scaling constant and is defined in the next subsection.

B. Notation

The *degree* of a node u, denoted as $d(u)$, is the number of its neighbors directly connected to it [17]. A node of degree zero is called an *isolated node*. The minimum node degree of a graph G is defined as $d_{min}(G) = \min_{u \in V(G)}\{d(u)\}$.

Throughout the paper, we will use standard mathematical notations concerning the asymptotic behavior of functions, i.e., $f(n) = o(g(n))$ or $f(n) \ll g(n)$ if $\frac{f(n)}{g(n)} \to 0$ as $n \to \infty$; $f(n) = O(g(n))$ if there exists a constant c and a value n_0 such that $f(n) \le c \cdot g(n)$ for all $n \ge n_0$ [18]. Symbols "o" and "O" always apply in the limiting case when $n \to \infty$. Define the notation $(\cdot)_+$ as $y_+ = y$ if $y \ge 0$ and $y_+ = 0$ if $y \le 0$. Define π_d $(d=1,2,3)$ as $\pi_1 = 1$, $\pi_2 = \pi$ and $\pi_3 = \frac{4}{3}\pi$.

III. Related works

Krishnamachari *et al.* [6] presented some examples in wireless ad hoc networks where a phase transition phenomenon exists, such as connectivity, coordination and probabilistic flooding for route discovery. The authors then pointed out the significance of understanding phase transitions. In [7], Krishnamachari *et al.* investigated three distributed configuration tasks in wireless multi-hop networks, viz., partition into coordinating cliques, Hamiltonian cycle formation and conflict-free channel allocation. The authors showed that these tasks undergo phase transitions with respect to transmission range, and argued that phase transition analysis is useful in self-configuration of wireless networks. In [14], Ravelomanana showed that network coverage is subject to abrupt phase transition in 3-dimensional wireless sensor networks. In [10], Aspnes *et al.* exhibited with simulation evidence the phase transition phenomenon for localizability of wireless sensor networks.

In [8], Goel *et al.* proved that all monotone properties [8], [19] in random geometric graphs have a sharp phase transition width, $\delta(n,\alpha)$, which is bounded by

$$\delta(n,\alpha) \le \bar{\delta}(n,\alpha) = \begin{cases} O(\log^{1/2}\frac{1}{\alpha}/n^{1/2}), & d=1 \\ O(\log^{3/4} n/n^{1/2}), & d=2 \\ O(\log^{1/d} n/n^{1/d}), & d \ge 3 \end{cases} \tag{3}$$

Because Eq. 3 are upper bounds for all monotone properties, they may be quite conservative for some specific monotone properties.

Han *et al.* [5] pointed out that because the results in [8] were derived for a *generic* monotone property, they may be further sharpened for certain *specific* monotone graph properties such as connectivity. The authors improved the results of Goel *et al.* for the property of network connectivity in one and two dimensional spaces, and derived the phase transition width of connectivity for large n, i.e.,

$$\delta_1(n,\alpha) = \begin{cases} \frac{C(\alpha)}{n} + o(n^{-1}), & d=1 \\ \frac{C(\alpha)}{2}\sqrt{\frac{1}{\pi n \log n}}(1+o(1)), & d=2 \end{cases} \tag{4}$$

where $C(\alpha) = \log(\frac{\log\alpha}{\log(1-\alpha)})$. The results are much sharper than the results given in Eq. 3, which indicates that the results in Eq. 3 are quite conservative for a specific monotone property.

IV. Main results

In this section, we present the main results on the phase transition width $\delta_k(n,\alpha)$ for k-connectivity $(k \in \mathbb{N})$ in d-dimensional space ($d=1,2,3$). We have ignored the boundary effect in our derivation for $d=2,3$.

First, we present the following lemma which will be used in the proof of later theorem.

Lemma 1. *Consider $G(n,r(n))$ in $\Re^d$ $(d=2,3)$, an integer $k>0$ and a real number $\omega \in \Re$. Let $\xi(k,n,r(n))$ be the expected number of nodes with degree k. Let $r(n)$ satisfy*

$$r(n) = r_n(\omega) = \left(\frac{F(n,k)+\omega}{\pi_d n}\right)^{\frac{1}{d}}, \tag{5}$$

where

$$F(n,k) := \log n + (k-1)\log(\log n) - \log(k-1)!. \tag{6}$$

Then, ignoring the boundary effect, the following holds:

$$\xi(k-1,n,r_n(\omega)) = e^{-\omega}, \quad k=1. \tag{7}$$

$$\lim_{n\to\infty}\xi(k-1,n,r_n(\omega)) = e^{-\omega}, \quad k>1. \tag{8}$$

Proof: Using the Poisson approximation (see subsection II-A) and ignoring the boundary effect, the probability that a randomly chosen node i $(i=1,2,...,n)$ has m $(m \ge 0)$ neighbors, denoted as $p_i(m)$, is given by

$$p_i(m) = \frac{\left(n\pi_d r^d(n)\right)^m}{m!}e^{-n\pi_d r^d(n)}, \quad d=2,\ 3. \tag{9}$$

Therefore, using *Palm Theorem* [13, pp. 20, 155], and ignoring the boundary effect, it can be shown that

$$\xi(k-1,n,r(n)) = n\cdot p_i(k-1) = n\cdot\frac{(n\pi_d r^d(n))^{k-1}}{(k-1)!}e^{-n\pi_d r^d(n)}. \tag{10}$$

For $k=1$, substituting Eq. 5 into Eq. 10, we have

$$\xi(0,n,r_n(\omega)) = n\cdot\exp\left(-n\cdot\pi_d\cdot\frac{\log n+\omega}{n\pi_d}\right) = e^{-\omega}.$$

For $k>1$, substituting Eq. 5 into Eq. 10, we have

$$\begin{aligned}
&\lim_{n\to\infty}\xi(k-1,n,r_n(\omega))\\
=&\lim_{n\to\infty}\left(n\cdot\frac{(F(n,k)+\omega)^{k-1}}{(k-1)!}\cdot\frac{1}{n}\cdot\frac{(k-1)!}{(\log n)^{k-1}}\cdot e^{-\omega}\right)\\
=&\ e^{-\omega}\lim_{n\to\infty}\left(1+\frac{(k-1)\log(\log n)}{\log n}\right.\\
&\left.-\frac{\log(k-1)!}{\log n}+\frac{\omega}{\log n}\right)^{k-1}\\
=&\ e^{-\omega}.
\end{aligned}$$

■

By the above lemma, the following theorem can be obtained.

Theorem 1. *Consider $G(n,r(n))$ in $\Re^d$ $(d=1,2,3)$, an integer $k>0$ and a real number $\alpha\in(0,1)$. Ignore the boundary effect except when $d=1$. Then, for large n,*

$$r_k(n,\alpha) = \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} - \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\left(F(n,k)\right)^{\frac{d-1}{d}}}. \quad (11)$$

Proof:

Case 1 ($d=2,\ 3$). Let $\gamma_k(n)$ (respectively, $\tau_k(n)$) denote the minimum transmission range at which a random geometric graph $G(n,r(n))$ is k-connected (respectively, has minimum node degree k). Penrose has proved the following results:

Proposition 1 (Theorem 1.1 in [15]). *Consider $G(n,r(n))$ in $\Re^d$ $(d\geq 2)$. Given any integer $k>0$,*

$$\lim_{n\to\infty} Pr\{\gamma_k(n)=\tau_k(n)\}=1. \quad (12)$$

Proposition 2 (Theorem 1.2 in [15]). *Consider $G(n,r(n))$ in $\Re^d$ $(d\geq 1)$. Let $\omega\in\Re$; let $\xi(k,n,r(n))$ be the expected number of nodes with degree k. Given any integer $k>0$ and $(r_n)_{n\geq 1}$ satisfying the following condition*

$$\lim_{n\to\infty}\xi(k-1,n,r_n)=e^{-\omega}, \quad (13)$$

then

$$\lim_{n\to\infty} Pr\{\tau_k(n)\leq r_n\}=\exp(-e^{-\omega}). \quad (14)$$

For any $\omega\in\Re$ and any integer $k>0$, Proposition 1 and Proposition 2 immediately yield

$$\lim_{n\to\infty} Pr\{\gamma_k(n)\leq r_n\}=\exp(-e^{-\omega}). \quad (15)$$

Eq. 15 gives the limiting distribution function of $\gamma_k(n)$. Using Eq. 15 and Lemma 1, it can be shown that

$$\lim_{n\to\infty} P_k(n,r_n(\omega))=\exp(-e^{-\omega}). \quad (16)$$

For each $x\in\Re$, define a $[0,1]$-valued sequence $\{\sigma_n(x), n=1,2,3...\}$ by

$$\sigma_n(x)=\min\left(1,\left(\frac{F(n,k)+x}{\pi_d n}\right)_+^{\frac{1}{d}}\right),\ n=1,2,... \quad (17)$$

Because for any fixed integer $k>0$ and $x\in\Re$, $\frac{F(n,k)}{n}\to 0$ as $n\to\infty$, there exists a finite integer $N(k,x)$ such that

$$0<\left(\frac{F(n,k)+x}{\pi_d n}\right)^{\frac{1}{d}}<1,\quad \forall n>N(k,x).$$

Hence, we have

$$\sigma_n(x)=\left(\frac{F(n,k)+x}{\pi_d n}\right)^{\frac{1}{d}},\quad \forall n>N(k,x). \quad (18)$$

Therefore, from Lemma 1 and Eq. 16, we have

$$\lim_{n\to\infty} P_k(n,\sigma_n(x))=\exp(-e^{-x}). \quad (19)$$

Now fix x in $\Re$, from Eq. 19, we can obtain that for each $\varepsilon>0$, there exists a finite integer $N(\varepsilon,k,x)$ such that

$$\left|P_k(n,\sigma_n(x))-\exp(-e^{-x})\right|<\varepsilon,\quad \forall n>N(\varepsilon,k,x). \quad (20)$$

It can be easily found that the mapping $\Re\to\Re_+ : x\to\exp(-e^{-x})$ is strictly monotonically increasing and continuous with $\lim_{x\to-\infty}\exp(-e^{-x})=0$ and $\lim_{x\to\infty}\exp(-e^{-x})=1$. Therefore, for each $\alpha\in(0,1)$, there exists a unique value of x in $\Re$, denoted as x_α, such that $\exp(-e^{-x_\alpha})=\alpha$. In fact, from the equality $\exp(-e^{-x_\alpha})=\alpha$, we have

$$x_\alpha=-\log(-\log\alpha). \quad (21)$$

Hence, fixing x in $\Re$ is equivalent to fixing α in $(0,1)$. Now fix α in the interval $(0,1)$, and let ε be sufficiently small such that $0<2\varepsilon<\alpha$ and $\alpha+2\varepsilon<1$. Then applying Eq. 20 with $x=x_{\alpha+\varepsilon}$ and $x=x_{\alpha-\varepsilon}$ respectively, we have

$$\left|P_k(n,\sigma_n(x_{\alpha+\varepsilon}))-\exp(-e^{-x_{\alpha+\varepsilon}})\right|<\varepsilon,\ \forall n>N(\varepsilon,k,x_{\alpha+\varepsilon}) \quad (22)$$

and

$$\left|P_k(n,\sigma_n(x_{\alpha-\varepsilon}))-\exp(-e^{-x_{\alpha-\varepsilon}})\right|<\varepsilon,\ \forall n>N(\varepsilon,k,x_{\alpha-\varepsilon}). \quad (23)$$

We always assume that n is sufficiently large when necessary. In the rest of this case 1, we assume that $n>N(\varepsilon,k,\alpha)$ with

$$N(\varepsilon,k,\alpha)=\max\{N(k,x_\alpha),\ N(\varepsilon,k,x_{\alpha+\varepsilon}),\ N(\varepsilon,k,x_{\alpha-\varepsilon})\},$$

where $N(k,x_\alpha)$ represents the finite integer above which Eq. 18 holds.

Since $\exp(-e^{-x_{\alpha\pm\varepsilon}})=\alpha\pm\varepsilon$, it can be readily obtained from Eq. 22 and Eq. 23 that

$$\alpha<P_k(n,\sigma_n(x_{\alpha+\varepsilon}))<\alpha+2\varepsilon$$

and

$$\alpha-2\varepsilon<P_k(n,\sigma_n(x_{\alpha-\varepsilon}))<\alpha.$$

According to the definition of $r_k(n,\alpha)$, we have $P_k(n,r_k(n,\alpha))=\alpha$. Hence, from the last two inequalities, it follows that

$$P_k(n,\sigma_n(x_{\alpha-\varepsilon}))<P_k(n,r_k(n,\alpha))<P_k(n,\sigma_n(x_{\alpha+\varepsilon})).$$

Because of the strict monotonicity of the mapping $r(n)\to P_k(n,r(n))$, we have

$$\sigma_n(x_{\alpha-\varepsilon})<r_k(n,\alpha)<\sigma_n(x_{\alpha+\varepsilon}). \quad (24)$$

Define $\eta(n,\alpha) := r_k(n,\alpha) - \sigma_n(x_\alpha)$, then it can be obtained from Eq. 24 that

$$\sigma_n(x_{\alpha-\varepsilon}) - \sigma_n(x_\alpha) < \eta(n,\alpha) < \sigma_n(x_{\alpha+\varepsilon}) - \sigma_n(x_\alpha). \quad (25)$$

For any fixed $k > 0$ and $x \in \Re$, it is clear that

$$\begin{aligned} &\lim_{n\to\infty} \frac{x}{F(n,k)} \\ = &\lim_{n\to\infty} \frac{x}{\log n + (k-1)\log(\log n) - \log(k-1)!} = 0. \end{aligned}$$

Hence, from Eq. 18, we have

$$\begin{aligned} \sigma_n(x) &= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{x}{F(n,k)}\right)^{\frac{1}{d}} \\ &= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{1}{d}\frac{x(1+o(1))}{F(n,k)}\right), as\ n \to \infty. \end{aligned}$$

Therefore, for any fixed $k > 0$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} &\sigma_n(x_{\alpha\pm\varepsilon}) - \sigma_n(x_\alpha) \\ = &\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \frac{x_{\alpha\pm\varepsilon} - x_\alpha}{F(n,k)d}(1+o(1)), as\ n \to \infty. \quad (26) \end{aligned}$$

Because Eq. 25 holds for all $n > N(\varepsilon, k, \alpha)$, it must be valid when $n \to \infty$ as well. And as $n \to \infty$, the small order part $o(1)$ in Eq. 26 goes to zero. Hence, from Eq. 25 and Eq. 26, we have

$$x_{\alpha-\varepsilon} - x_\alpha \leq \liminf_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}} \eta(n,\alpha) \right) \quad (27)$$

and

$$x_{\alpha+\varepsilon} - x_\alpha \geq \limsup_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}} \eta(n,\alpha) \right). \quad (28)$$

Because ε can be chosen to be arbitrarily small, and as stated earlier $x_\alpha = -\log(-\log\alpha)$ is a continuous and strictly monotonically increasing function of α for $\alpha \in (0,1)$, it can be shown that

$$\lim_{\varepsilon\downarrow 0}(x_{\alpha-\varepsilon} - x_\alpha) = \lim_{\varepsilon\downarrow 0}(x_{\alpha+\varepsilon} - x_\alpha) = 0. \quad (29)$$

Hence, from Eq. 27, Eq. 28 and Eq. 29, we have

$$\lim_{n\to\infty} \left(\frac{F(n,k)d}{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}} \eta(n,\alpha) \right) = 0. \quad (30)$$

Thus, given that Eq. 30 holds, it must be true that

$$\eta(n,\alpha) = o\left(\frac{\left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}}}{F(n,k)d} \right) = o\left(\frac{1}{d\,(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}} \right).$$

Hence, we have

$$\begin{aligned} r_k(n,\alpha) &= \sigma_n(x_\alpha) + \eta(n,\alpha) \\ &= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} \left(1 + \frac{x_\alpha(1+o(1))}{F(n,k)d}\right) \\ &\quad + o\left(\frac{1}{d\,(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}\right) \\ &= \left(\frac{F(n,k)}{\pi_d n}\right)^{\frac{1}{d}} - \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{d(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}. \end{aligned}$$

Case 2 ($d = 1$). When $k = 1$, Eq. 11 readily becomes Han *et al*'s result (see Eq. 4 in [5]), therefore, the result given in Theorem 1 is true for $d = 1$ and $k = 1$.

When $k > 1$, we shall prove the result based on the following proposition.

Proposition 3 (Theorem 15 in [20]). *Consider $G(n, r(n))$ in $\Re^d$ ($d = 1$). Let $\omega \in \Re$. Given any positive integer $k > 1$ and $r(n)$ satisfying*

$$r(n) = \frac{1}{n}(\log n + (k-1)\log(\log n) - \log(k-1)! + \omega),$$

then

$$\lim_{n\to\infty} P_k(n, r(n)) = \exp(-e^{-\omega}).$$

Based on the critical radius given in Proposition 3 and using the same technique as described in case 1, we can obtain

$$r_k(n,\alpha) = \frac{F(n,k)}{n} - \frac{\log\left(\log\left(\frac{1}{\alpha}\right)\right)(1+o(1))}{n},\ d = 1,$$

which agrees with the result given in Theorem 1 for $d = 1$ and $k > 1$. It is important to notice that the boundary effect affects neither the derivation of Han *et al*'s result in [5] nor the derivation of Theorem 15 in [20]. Hence, our result in this paper is not affected by the boundary effect when $d = 1$.

Combining cases 1 and 2, we have finally proved Theorem 1. Note that some parts of the proof used here are similar to the arguments used in [5]. ■

By Theorem 1, the phase transition width $\delta_k(n,\alpha)$ for large n can be derived in the following Corollary 1.

Corollary 1. *Consider $G(n, r(n))$ in $\Re^d$ ($d = 1, 2, 3$), an integer $k > 0$ and a real number $\alpha \in (0, \frac{1}{2})$. Ignore the boundary effect except when $d = 1$. Then, for large n*

$$\delta_k(n,\alpha) = \frac{\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)}{d\,(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}(1+o(1)). \quad (31)$$

Proof: From Theorem 1, for any fixed integer $k > 0$ and $d \in \{1,2,3\}$, $\delta_k(n,\alpha)$ for large n can be derived as

$$\begin{aligned} \delta_k(n,\alpha) &= r_k(n, 1-\alpha) - r_k(n,\alpha) \\ &= \frac{\log\left(\frac{\log\alpha}{\log(1-\alpha)}\right)}{d(\pi_d n)^{\frac{1}{d}}\,(F(n,k))^{\frac{d-1}{d}}}(1+o(1)). \end{aligned}$$

■

Comparing Eq. 4 and Eq. 31, we can see that when $k = 1$ and $d = 1, 2$, Eq. 31 reduces to Eq. 4, which appears in [5]. By Corollary 1, three further corollaries can also be obtained.

Corollary 2. *Adopt the same hypothesis as Corollary 1. Then, $\delta_k(n, \alpha)$ and $\delta_{k+1}(n, \alpha)$ satisfy*

$$\lim_{n\to\infty} \frac{\delta_{k+1}(n, \alpha)}{\delta_k(n, \alpha)} = 1. \tag{32}$$

Proof: From Corollary 1, for any fixed integer $k > 0$ and $d \in \{1, 2, 3\}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\delta_{k+1}(n, \alpha)}{\delta_k(n, \alpha)} \\
= &\lim_{n\to\infty} \left(\frac{\log n + (k-1)\log(\log n) - \log(k-1)!}{\log n + k\log(\log n) - \log k!} \right)^{\frac{d-1}{d}} \\
= &\lim_{n\to\infty} \left(\frac{1 + \frac{(k-1)\log(\log n)}{\log n} - \frac{\log(k-1)!}{\log n}}{1 + \frac{k\log(\log n)}{\log n} - \frac{\log k!}{\log n}} \right)^{\frac{d-1}{d}} = 1.
\end{aligned}$$

■

Remark. Corollary 2 means that at large n, the phase transition widths for k-connectivity and $(k+1)$-connectivity are approximately equal.

Corollary 3. *Adopt the same hypothesis as Corollary 1. Then, for large n, $\delta_k(n, \alpha)$ in $(j+1)$-dimensional space is larger than that in j-dimensional space, where $j = 1, 2$.*

Proof: Let $\delta_k^{(d)}(n, \alpha)$ denote the phase transition width of k-connectivity in d-dimensional space as we want to emphasize the dependence of $\delta_k(n, \alpha)$ on d. Let $j \in \{1, 2\}$. Then, from Corollary 1, we have

$$\frac{\delta_k^{(j+1)}(n, \alpha)}{\delta_k^{(j)}(n, \alpha)} = \frac{j}{j+1} \frac{(\pi_j)^{\frac{1}{j}}}{(\pi_{j+1})^{\frac{1}{j+1}}} \left(\frac{n}{F(n, k)} \right)^{\frac{1}{j(j+1)}} (1+o(1)). \tag{33}$$

For any fixed positive integer $k > 0$ and $j \in \{1, 2\}$, there exists a finite integer $N(k, j)$ such that

$$\frac{j}{j+1} \frac{(\pi_j)^{\frac{1}{j}}}{(\pi_{j+1})^{\frac{1}{j+1}}} \left(\frac{n}{F(n, k)} \right)^{\frac{1}{j(j+1)}} (1 + o(1)) > 1$$

for all $n > N(k, j)$. Hence, from Eq. 33, we have $\delta_k^{(j+1)}(n, \alpha) > \delta_k^{(j)}(n, \alpha)$ $(j = 1, 2)$ for large n. ■

Corollary 4. *Adopt the same hypothesis as Corollary 1. Then, for large n, the variation of $\delta_k(n, \alpha)$ with α is separable from the variation with n, k and d in the sense that for some functions T and Y, there holds $\delta_k(n, \alpha) = T(\alpha) \cdot Y(n, k, d)$.*

Proof: Let $\delta_k^{(d)}(n, \alpha)$ denote the phase transition width of k-connectivity in d-dimensional space as we want to emphasize the dependence of $\delta_k(n, \alpha)$ on d. We assume that n is large enough such that the small order part $o(1)$ in Eq. 31 can be ignored. Then, for any fixed n, k and d, we have

$$\begin{aligned}
\delta_k^{(d)}(n, \alpha) &\approx \log\left(\frac{\log \alpha}{\log(1-\alpha)} \right) \cdot \frac{1}{d\,(\pi_d n)^{\frac{1}{d}} \,(F(n, k))^{\frac{d-1}{d}}} \\
&= T(\alpha) \cdot Y(n, k, d).
\end{aligned}$$

■

Remark. For any fixed n, k and d, the term $Y(n, k, d)$ is fixed and is independent of α. Hence, $Y(n, k, d)$ does not change as α varies given that n, k and d are fixed. Therefore, the variation of $\delta_k^{(d)}(n, \alpha)$ with α has the same functional dependence irrespective of n, k and d, save for a scaling constant defined by these latter variables.

V. SIMULATIONS

In this section, we report simulations conducted to verify the theoretical analysis. Note that in [5] and [8], no simulation results are presented. Thus there is no verification for the analytical results in [5], [8].

We programmed a tool in C++ for the simulations. In the simulations, we consider that a total of n nodes are randomly and uniformly distributed in a unit cube $[0, 1]^d$ $(d = 1, 2, 3)$ and all nodes have the same transmission range. We have used the *toroidal distance metric* [1] to remove the impact of the boundary effect on the simulation results for $d = 2,\ 3$. Because simulations become very computationally intensive and time consuming for $k > 3$ and large values of n, we limited k to 3 and n to 2000 in the simulations.

Fig. 1 shows the analytical results and the simulation results for $\delta_1(n, \alpha)$ in $\Re^d$ $(d = 1, 2, 3)$. The value of n is varied between 100 and 2000, α is set to two typical values, i.e., 0.4 (close to 0.5) and 0.05 (close to 0). When calculating the analytical results by Eq. 31, the small order part is omitted, i.e., $o(1)$ in the term $(1 + o(1))$ is ignored. We can see that for each $d \in \{1, 2, 3\}$, $\delta_1(n, \alpha)$ decreases as n grows, which is consistent with Corollary 1. It is obvious that there is significant discrepancy between the analytical results and the simulation results for small values of n (e.g., $n = 100, 200$). This is because the small order part $o(1)$ in the analytical results is significant when n is small. However, the small order part $o(1)$ goes to zero as n goes to infinity. We can also see that $\delta_1(n, \alpha)$ is larger for $d = 3$ than that for $d = 2$, and similarly, $\delta_1(n, \alpha)$ is larger for $d = 2$ than that for $d = 1$, which verifies Corollary 3. This means that in a higher dimensional network, more transmission power is needed in order to make the probability that the network is k-connected transit from almost zero to almost one.

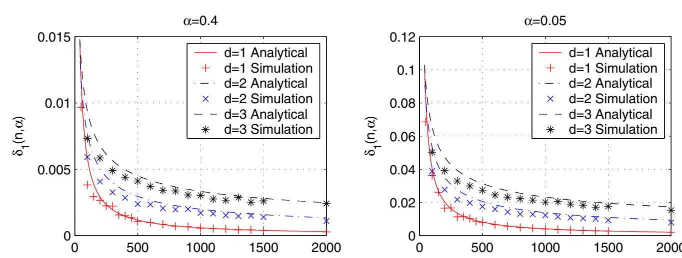

Fig. 1. $\delta_1(n, \alpha)$ *versus* n in d-dimensional space $d = 1, 2, 3$. The value of α is fixed in each scenario.

Fig. 2 shows the analytical results and the simulation results for $\delta_k(n, \alpha)$ $(k = 1, 2, 3)$ in 2-dimensional space. Other settings are the same as in Fig. 1. We can see that when $d = 2$, $\delta_k(n, \alpha)$ decreases as n increases, which is also consistent with Corollary 1. The figure also indicates that the difference between $\delta_k(n, \alpha)$ and $\delta_{k+1}(n, \alpha)$ becomes smaller as n gets larger. This means that $\delta_k(n, \alpha) \approx \delta_{k+1}(n, \alpha)$ when n is large enough, which is consistent with Corollary 2. In other words, the energy required for making the probability that the network is k-connected increase from almost zero to almost one is approximately the same as the energy required for making the probability that the network is $(k+1)$-connected increase from almost zero to almost one.

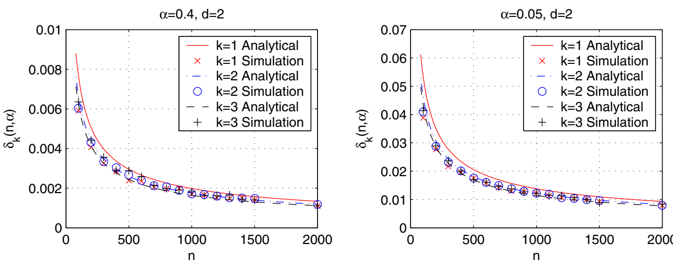

Fig. 2. $\delta_k(n, \alpha)$ *versus* n in 2-dimensional space. The value of α is fixed in each scenario.

Fig. 3 shows the dependence of $\delta_k(n, \alpha)$ on α given that n, k and d are fixed. We can see that the variation of $\delta_k(n, \alpha)$ with α has the same functional dependence irrespective of n, k and d. This verifies Corollary 4. We can also see that for fixed n, k and d, $\delta_k(n, \alpha)$ decreases as α increases. In addition, the discrepancy between the analytical results and the simulation results becomes significant when α is very small. The reason for this is that the small order part $o(1)$ in the analytical results becomes significant when α is very close to zero.

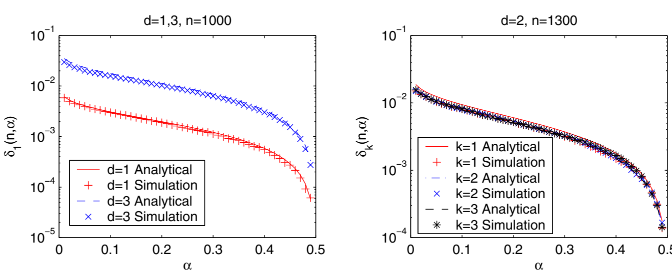

Fig. 3. $\delta_k(n, \alpha)$ *versus* α in d-dimensional space $d = 1, 2, 3$. The value of n is fixed in each scenario.

VI. Conclusion and Future Work

In this paper, we investigated the phase transition behavior of k-connectivity in wireless multi-hop networks. For large n, we derived a generic analytical formula for calculating the phase transition width of k-connectivity for any $k \in \mathbb{N}$ in d-dimensional space ($d = 1, 2, 3$). We also presented simulations conducted to verify the theoretical analysis. These results are very useful in the design, self-configuration, and transmission power control in wireless sensor/ad hoc networks.

One direction of our future work is to investigate the phase transition width of k-connectivity considering the boundary effect. Another direction is to study the size of n above which the results can be appropriately applied.

A more extensive version of these results including both theoretical justification and simulations can be found in [21].

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