This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

Network Capacity Maximization Using Route Choice and Signal Control With Multiple OD Pairs

Shangbo Wang, *Member, IEEE*, Changle Li, *Senior Member, IEEE*, Wenwei Yue, and Guoqiang Mao, *Fellow, IEEE*

***Abstract*— In this paper, we investigate a hybrid dynamical system which incorporates flow swap process, green-time proportion swap process, and flow divergence for a general network with multiple Origin-Destination (OD) pairs and multiple routes, where flow swap process is specified in which traffic swaps from more costly to less costly input links, green-time proportion swap process is specified in which green time at each intersection swaps from less pressurized stages to more pressurized stages, flow may diverge at each intersection from one OD pair to other OD pairs. Unlike the dynamical system model, where bottleneck delays need to be intentionally constructed to yield the equilibrium flow vector and green-time proportion vector, we propose a novel control policy to fill the gap by only adjusting the green-time proportion vector. We derive a sufficient condition for the existence of equilibrium of the dynamical system under the mild constraints that 1) the travel cost function and stage pressure function should be continuous functions and 2) the flow and green-time proportion swap processes project all flow and green-time proportion vectors on the boundary of the feasible region onto itself. We derive the condition of unique equilibrium for fixed green-time proportion vector and show that with varying green-time proportion vector, the set of equilibria is a compact, non-convex set, and with the same partial derivative of travel cost function with respect to the flow and green-time proportion vectors. Finally, we prove the stability of the proposed dynamical system by using Lyapunov stability analysis.**

***Index Terms*— Hybrid dynamical system, flow swap process, green-time proportion swap process, Laypunov stability analysis.**

I. Introduction

TRAFFIC demand is ever increasing due to urbanization and rapid growth of population. Therefore, how to economically utilize the spatial-temporal resources of roads and traffic signal settings to maximize road network capacity becomes an important issue. In the literature, capacity of a transport network is defined as the maximum link dynamical route flow that the network can accommodate under a given travel demand. Without consideration of control policies such as route choice and traffic signal control, the capacity maximization problem becomes a max-flow min-cut problem that can be directly drawn from network flow and graph theory [2], [3]. In practice, however, the maximum throughput between one pair of OD (Origin-Destination) nodes may deviate from sum of the flows on the bottlenecks because of traffic signal control applied on each intersection and drivers' route choice.

To achieve an optimal throughput, there are some recent research efforts studying the impact on network capacity posed by optimizing isolated or coordinated traffic signals, and driver's route [1], [4]–[7]. They mainly aim at answering the following questions: by a given traffic assignment model, 1) does the Wardrop equilibrium exist? 2) is the Wardrop equilibrium unique? 3) is the dynamical system stable? To this end, Smith et al. proposed P0 control policy [8] and proved that by using P0 control policy on a simple network, Wardrop equilibrium can be found by intentionally constructing bottleneck delay on each link, that is, the maximum network throughput can be achieved at a quasi-dynamic user equilibrium [1], [4]. Xiao and Lo investigated the interaction between adaptive traffic control and day-to-day route choice adjustments of travelers in a dynamical system setting [5]. Yang *et al.* modeled the capacity and level of service, and investigated the additional demand tolerance of urban networks [7]. Le *et al.* proposed a decentralized traffic signal control policy to optimize the throughput on a very simple one junction network [6]. Smith proposed P0 control policy and showed that under natural conditions a simple network is stable under this policy, and this policy can achieve capacity maximization [9]. Chen and Kasikitwiwat proposed two approaches for assessing the value of capacity flexibility based on the concept of reserve capacity and demand changes, respectively. Chen and Kasikitwiwat considered the capacity flexibility influenced by user's free choice of routes and destinations [10]. Chiou presented a new solution that simultaneously maximizes increase in travel demands and minimizes link capacity expansion investment [11].

To the authors' knowledge, almost all existing research efforts focus on seeking an optimal control policy to optimize throughput by considering one or multiple OD (Origin-Destination) pair, without consideration of the flow divergence among different OD pairs caused by the turning matrix. Flow divergence is defined by the traffic switching to another link/route midway in its route when the routes of multiple OD pair intersect each other (see Fig. 1 and 2). P0 control

Manuscript received September 4, 2018; revised January 23, 2019; accepted March 31, 2019. This work was supported in part by the National Natural Science Foundation of China under Grant U1801266 and Grant 61571350, in part by the Key Research and Development Program of Shaanxi under Contract 2018ZDXM-GY-038 and Contract 2018ZDCXL-GY-04-02, and in part by The Youth Innovation Team of Shaanxi Universities. The Associate Editor for this paper was S. Siri. *(Corresponding author: Changle Li.)*

S. Wang and G. Mao are with the School of Computing and Communications, University of Technology Sydney, Sydney, NSW 2007, Australia (e-mail: shangbo.wang@uts.edu.au; guoqiang.mao@uts.edu.au).

C. Li and W. Yue are with the State Key Laboratory of Integrated Services Networks, Xidian University, Xi'an 710126, China (e-mail: clli@mail.xidian.edu.cn; yuewenwei123@foxmail.com).

Digital Object Identifier 10.1109/TITS.2019.2909281

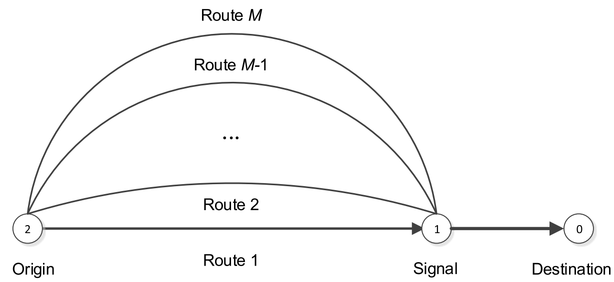

Fig. 1. A *M* route signalized network proposed by Smith. Only one OD pair was considered.

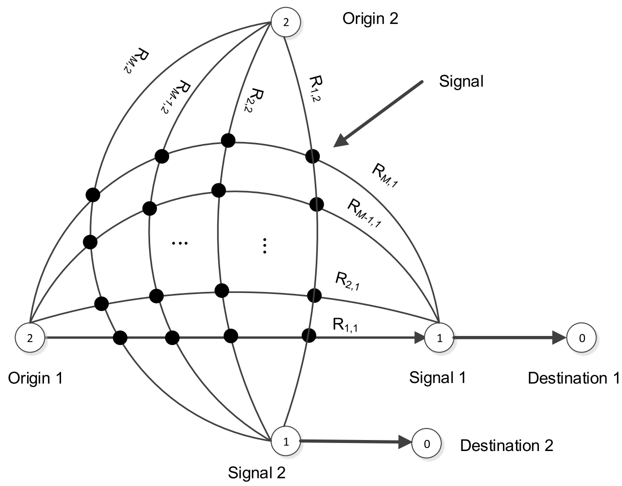

Fig. 2. A signalized network with two OD pairs, where each intersection is equipped with traffic signal.

policy proposed by Smith [1], [4], [8] has been verified to outperform other control policies such as equi-saturation policy [12] because it can maximize capacity at a quasi-dynamic user equilibrium on a simple network, which consists of multiple OD pairs with a group of routes and does not have any flow divergence at each route. In an urban network, there may be multiple OD pairs, each of which is connected via multiple routes that may interwine with those affiliated to different OD pairs (Fig. 2). In the presence of flow divergence among different OD pairs, the route flow swap process and traffic signal assignment process become much more intricate compared to those for a simple network because the traffic signal assignment process can also occur between different OD pairs, rather than just different routes for the same OD pair; the route flow swap process should take into account each link's cost on the same route. In summary, no existing work investigated the conditions for the existence and uniqueness of equilibrium and stability of the dynamical systems when flow swap process, green-time proportion swap process and flow divergence among different OD pairs are simultaneously considered.

This paper aims at addressing the aforementioned theoretical gap and extending the existing traffic assignment model to a more realistic scenario. Specifically, the main contributions of this paper are as follows:

- We propose an objective function for a general network with multiple OD pairs and multiple routes to find the equilibrium link flow and green-time proportion vector;
- A novel control policy is proposed for the general network to maximize the network capacity with a steady demand and vertical queue and the superiority of the novel control policy over P0 control policy is shown;
- Based on P0 control policy and the novel control policy, a hybrid traffic assignment model incorporating the flow swap process, green-time proportion swap process and flow divergence is proposed for the general network;
- The conditions for the existence and uniqueness of Wardrop equilibrium, and the characteristic of the set of equilibria are given and the stability of the dynamical system is proved by using Lyapunov stability analysis.

The rest of the paper is outlined as follows: Section II reviews related work. Section III introduces the general network and the novel traffic assignment model incorporating flow divergence. Section IV provides the hybrid dynamical system by incorporating the flow swap process and green-time proportion swap process, shows the condition for existence and uniqueness of Wardrop equilibrium, the characteristic of the set of equilibria, and proves the stability of the dynamical system. Section V provides some numerical results to verify our theoretical findings. Section VI concludes the study.

II. Related Work

Smith et al. presented a more special dynamical models of day-to-day re-rerouting and green-time response. They showed that with responsive P0 control policy, a simple network can achieve the maximum capacity at a quasi-dynamic user equilibrium, and queues and delays in the simple network remain bounded in natural dynamical evolution [4], [9]. They showed in [1] that applying P0 control policy in a simple network, the convergence of stage green-time can be guaranteed in the vertical queuing case. Xiao and Lo addressed the equilibrium stability and convergence of the proposed joint dynamic traffic system by analyzing the Jacobian matrix associated with each fixed point in a simple network [5]. Smith introduced the restrictive proportional adjustment process (RPAP) for route swaps, which is more restrictive for defining routes compared to the original proportional adjustment process (PAP) proposed by [13]. Yang *et al.* adopted the model of equilibrium trip distribution/assignment with variable destination costs (ETDA-VDC) to assign the flow on each route when the maximum O-D flows are obtained. They proposed a bi-level optimization model where the upper-level sub-problem aims at maximizing the sum of total trip generations and the lower-level sub-problem is the flow assignment. They considered the case of multiple OD pairs. However, the signal control and interference among multiple OD pairs were not considered [7]. Tassiulas and Ephremides proposed a decentralized signal control policy, which adopted the BackPressure scheme [14] to allocate the proportion of the common cycle length to each phase [6]. The main objective of that paper is to prove the stability (limited queue length) of the proposed signal control policy. Route choice was not taken into account and capacity maximization at equilibrium was not investigated. Chen and Kasikitwiwat provided a quantitative assessment of capacity flexibility for the passenger transportation network using bi-level network capacity models [10]. Variations in demand pattern and volume were allowed in that paper. The utilized bi-level optimization model was similar to that proposed in [7]. Signal control was not considered. Chiou aimed at optimizing traffic assignment by minimizing travelers' delay and maximizing reserve capacity [11] using the TRANSYT model [15]. Nie proposed a class of bush-based algorithms for the user equilibrium traffic assignment problem, which is in fact a variant of Beckmann's seminal formulation [16], [17]. Mounce and Carey proposed the route swap algorithm that the swap rate is proportional to the flow on the more costly route multiplied by the cost difference between two routes, and proved that with the proposed route swap algorithm, the dynamical system can converge to a global equilibrium [18].

Cascetta *et al.* formulated the Global Optimization of signal Settings (GOSS) problem, where the signal parameters are calculated by searching among all the feasible signal setting vectors. The authors proposed a new descent direction to solve the non-convex objective function [19]. Cascetta and Sforza incorporated signal setting parameters, path choice and user behaviour into the network design model, and proposed an iterative algorithm for the computation of the optimum signal setting parameters with an equilibrium flow pattern [20].

The aforementioned review did not investigate the capacity maximization at equilibrium by simultaneously considering flow swap and green-time proportion swap with flow divergence among different OD pairs.

III. Novel Traffic Assignment Model in a Realistic Network

For simplicity, we firstly introduce the novel traffic assignment model in a simple network with only two OD pairs, each of which has two routes. Then, we will extend the traffic assignment model to a generalized form with multiple OD pairs, each of which has M routes with $M > 2$. We use the definition of capacity maximization given in [1], which can be described as

Definition 1: Suppose P is a control policy. "The control policy P maximizes network capacity" means that: for a network with a rigid or steady demand and traffic signals; if there is a route-flow vector $\boldsymbol{X}$ which meets the given inelastic demand and is within the capacity region of the given network, then there exist a route-flow vector $\boldsymbol{X^*}$, delay vector $\boldsymbol{b^*}$, green-time proportion vector $\boldsymbol{G^*}$, which meet the given inelastic demand, and is within the capacity limitations of the given network. Furthermore, $\boldsymbol{X^*}$, $\boldsymbol{b^*}$ and $\boldsymbol{G^*}$ fulfill

1) $\boldsymbol{X^*}$ is a Wardrop equilibrium when the delay vector, green-time vector is $\left[\boldsymbol{b^*}\ \boldsymbol{G^*}\right]$ and
2) $\boldsymbol{G^*}$ satisfies the control policy P when the delay vector, route-flow vector is $\left[\boldsymbol{b^*}\ \boldsymbol{X^*}\right]$.

We give the definition of reserve capacity at equilibrium by

Definition 2: The reserve capacity at equilibrium is defined by the maximum possible increase in traffic demand that can be accommodated in the network considering (optimum) traffic signal control at each intersection and drivers' route choice behavior [21].

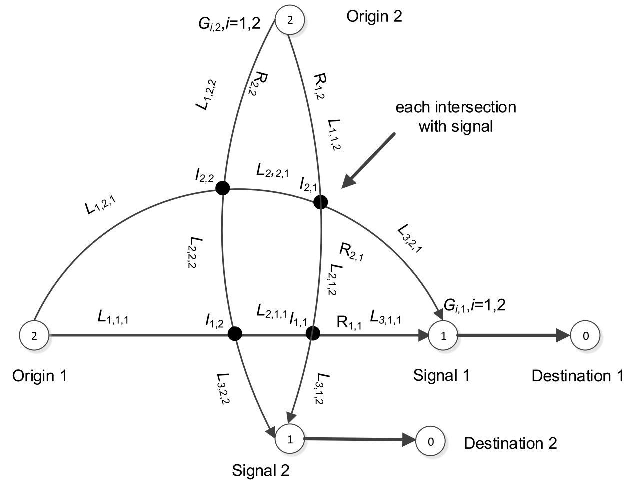

Fig. 3. A signalized network with two OD pairs with two routes each. There is traffic signal at each intersection.

A. Equilibrium in a Simple Network Without Signal Control (With Given Proportion of Green-Time for Each Traffic Signal)

Consider a simple network with two OD pairs with two routes each in a steady quasi-dynamic state with vertical queue (Fig. 3). The vertical queue assumes an idealized scenario for analytic purpose that vehicles on a roadway stack up upon one another at the point where congestion begins, rather than backing up along the length of roadway [22]. The vertical queue was used in [1], [4], and [8]. The definition of "steady quasi-dynamic" can be found in [1]. The symbol $R_{n,m}$ in Fig. 3 represents the n-th route of the m-th OD pair, where $n, m = 1, 2$.

Let us define $C_{n,m}$, $L_{n,m}$, $s_{n,m}$, $X_{n,m}$, $b_{n,m}$, $G_{n,m}$, $I_{n,n'}$ as the free-flow travel cost via $R_{n,m}$, the length of $R_{n,m}$, the saturation flow at $R_{n,m}$, the flow on $R_{n,m}$, the bottleneck delay of $R_{n,m}$ at the merging point (Signal 1 and 2 in Fig. 3), the green-time proportion assigned to $R_{n,m}$ at the merging point and the intersection between the n-th route of OD pair 1 and the n'-th route of OD pair 2, respectively. Let $G_{l,I_{n,n'}}$ and $b_{l,I_{n,n'}}$ be the green-time proportion and the bottleneck delay assigned to the l-th stage of the intersection $I_{n,n'}$, respectively. The unit of $C_{n,m}$, $b_{n,m}$ and $b_{l,I_{n,n'}}$ are seconds and the unit of $s_{n,m}$ and $X_{n,m}$ are vehicles per second. The saturation flow at $R_{n,m}$ denotes the capacity of $R_{n,m}$ at the critical density ρ_c if the green-time proportion of that stage is one for the traffic signals at all intersections and the merging point. The symbol $L_{l,n,m}, l = 1, 2, 3; n, m = 1, 2$ is defined by the l-th link for the n-th route of the m-th OD pair, $X_{l,n,m}$ is defined by the flow on the link $L_{l,n,m}$, $p_{(l,n,m),(l',n',m')}$ is defined by the turning rate from $L_{l,n,m}$ to $L_{l',n',m'}$.

Let us define $\boldsymbol{F}$, $\boldsymbol{D}$ and $\boldsymbol{S}$ by the feasible, demand and supply sets, respectively, then we have

$$\boldsymbol{F} = \left\{ \begin{pmatrix} G_{k,m} \\ G_{l,I_{n,n'}} \\ n, n' = 1, 2 \\ k, l = 1, 2 \\ m = 1, 2 \end{pmatrix} : \begin{array}{c} G_{1,m} + G_{2,m} = 1, \\ G_{1,I_{n,n'}} + G_{2,I_{n,n'}} = 1, \\ G_{k,m} \geq 0, G_{l,I_{n,n'}} \geq 0 \end{array} \right\} \quad (1)$$

Suppose for convenience that $s_{2,m} > s_{1,m}$ and $C_{2,m} > C_{1,m}$ for $m = 1, 2$; the other scenarios can be readily accommodated by our analysis. Let us define T_1 and T_2 as fixed demands at the origins 1 and 2, respectively. We firstly assume that there is no flow divergence at each intersection and no traffic signal coordination, then $p_{(l,n,m),(l',n',m')} = 1$ for $l' = l+1$, $(n, m) = (n', m')$ and $p_{(l,n,m),(l',n',m')} = 0$ for $(n, m) \neq (n', m')$.

Then, $\boldsymbol{D}$ and $\boldsymbol{S}$ can be obtained by

$$\boldsymbol{D} = \left\{ \begin{pmatrix} X_{n,m} \\ m, n = 1, 2 \end{pmatrix} : \begin{matrix} X_{1,m} + X_{2,m} = T_m \\ X_{n,m} \geq 0 \end{matrix} \right\} \tag{2}$$

and

$$\boldsymbol{S} = \left\{ \begin{pmatrix} X_{k,m} \\ G_{k,m} \\ G_{l,I_{n,n'}} \\ n, n' = 1, 2 \\ l, m = 1, 2 \\ k = 1, 2 \end{pmatrix} : \begin{matrix} X_{k,m} \leq \\ \min \begin{pmatrix} s_{k,m} G_{k,m} \\ s_{k,m} G_{m,I_{k,n'}} \end{pmatrix} \end{matrix} \right\} \tag{3}$$

From (2) and (3), the feasible demand set for T_m with $m = 1, 2$ can be determined by

$$\begin{aligned} T_m &\leq \min \begin{pmatrix} s_{1,m} G_{1,m} \\ s_{1,m} G_{m,I_{1,n'}} \end{pmatrix} + \min \begin{pmatrix} s_{2,m} G_{2,m} \\ s_{2,m} G_{m,I_{2,n'}} \end{pmatrix} \\ &\leq s_{1,m} G_{1,m} + s_{2,m} G_{2,m} \leq s_{2,m} \end{aligned} \tag{4}$$

and

$$T_1 + T_2 \leq \min \begin{bmatrix} s_{2,1} + s_{2,2} \\ \max(s_{1,1}, s_{1,2}) + \max(s_{2,1}, s_{2,2}) \\ \max(s_{1,1}, s_{2,2}) + \max(s_{2,1}, s_{1,2}) \end{bmatrix} \tag{5}$$

No green time proportion in $\boldsymbol{F}$ can be found to satisfy the route flow in $\boldsymbol{S}$ if T_1 and T_2 are not in the feasible demand set given by (4) and (5).

By the definition of Wardrop equilibrium that no traveler can reduce his travel cost by unilaterally changing routes [23], equilibrium flow on each route is obtained by solving the following equilibrium problem:

$$\begin{aligned} &\min_{\boldsymbol{X}} \sum_a \int_0^{X_a} t_a(\boldsymbol{X}, \boldsymbol{G})\, d\boldsymbol{X} \\ &st.\ (1), (2), (3) \end{aligned} \tag{6}$$

where $t_a(\boldsymbol{X}, \boldsymbol{G})$ is a strictly monotonically increasing travel cost function on the a-th route by the given flow vector $\boldsymbol{X}$ and the green time proportion vector $\boldsymbol{G}$ [17]. Taking first derivative of the objective function and setting it to zero, the equilibrium flow can be obtained by

$$\begin{aligned} t_{1,m}(\boldsymbol{X}^*, \boldsymbol{G}) &= t_{2,m}(\boldsymbol{X}^*, \boldsymbol{G}) \\ m &= 1, 2 \end{aligned} \tag{7}$$

which can be rewritten into the following equations:

$$\begin{aligned} C_{1,1} + b_{1,1} + b_{1,I_{1,1}} + b_{1,I_{1,2}} &= C_{2,1} + b_{2,1} + b_{1,I_{2,1}} + b_{1,I_{2,2}} \\ C_{1,2} + b_{1,2} + b_{2,I_{1,1}} + b_{2,I_{1,2}} &= C_{2,2} + b_{2,2} + b_{2,I_{2,1}} + b_{2,I_{2,2}} \end{aligned} \tag{8}$$

In [1], the bottleneck delays within a quasi-dynamic model are given by

$$b = Q/sG \tag{9}$$

where Q is the expected length of vertical queue at exit of the link and it is an explicit variable that is determined by the green time and the route flows. However, Q is hard to measure or estimate. One solution is to express the bottleneck delay as a function of the route flows, the green-time proportion and the cycle duration of traffic light.

The maximum bottleneck delay can be interpreted as the required time to *vacate* a queue caused by red signals. Then, in the simple network proposed in [1] with only one OD pair, the maximum bottleneck delay at one route fulfills the following relation [24]:

$$X(1 - G)\,\Delta T + Xb = sb \tag{10}$$

It follows that the maximum bottleneck delay can be expressed by

$$b = \frac{X(1 - G)\,\Delta T}{s - X} \tag{11}$$

where ΔT is the cycle duration. If $X = sG$, then $b = G\Delta T$, which is equal to the green time duration. An ever-increasing queue will be generated when $X > sG$. In the network proposed in Fig. 3, the maximum bottleneck delays at signal 1 and 2 can be expressed by

$$b_{n,m} = \frac{X_{n,m}(1 - G_{n,m})\,\Delta T}{s_{n,m} - X_{n,m}}; \quad m, n = 1, 2 \tag{12}$$

The maximum bottleneck delays at each intersection can be expressed by

$$b_{l,I_{n,n'}} = \frac{X_{n,l}\left(1 - G_{l,I_{n,n'}}\right)\Delta T}{s_{n',l} - X_{n,l}}; \quad l, n, n' = 1, 2 \tag{13}$$

Substituting (12) and (13) to (8), the equilibrium flow $X^*_{1,1}$, $X^*_{2,1}$, $X^*_{1,2}$ and $X^*_{2,2}$ can be determined by

$$\begin{aligned} &C_{1,1} + \frac{X^*_{1,1}\left[3 - \left(G_{1,1} + G_{1,I_{1,1}} + G_{1,I_{1,2}}\right)\right]\Delta T}{s_{1,1} - X^*_{1,1}} \\ &\quad = C_{2,1} + \frac{X^*_{2,1}\left[2 + \left(G_{1,1} - G_{1,I_{2,1}} - G_{1,I_{2,2}}\right)\right]\Delta T}{s_{2,1} - X^*_{2,1}} \\ &C_{1,2} + \frac{X^*_{1,2}\left[1 - \left(G_{1,2} - G_{1,I_{1,1}} - G_{1,I_{1,2}}\right)\right]\Delta T}{s_{1,2} - X^*_{1,2}} \\ &\quad = C_{2,2} + \frac{X^*_{2,2}\left[G_{1,2} + G_{1,I_{2,1}} + G_{1,I_{2,2}}\right]\Delta T}{s_{2,2} - X^*_{2,2}} \end{aligned} \tag{14}$$

Combining with (1), (2), (3), (12) and (13), the equilibrium flow and the bottleneck delays can be expressed by a function of $G_{n,m}$, $G_{l,I_{n,n'}}$, $C_{n,m}$, ΔT, T_m and $s_{n,m}$, where $C_{n,m}$, ΔT, T_m and $s_{n,m}$ are constants while $G_{n,m}$ and $G_{l,I_{n,n'}}$ are adjustable.

When flow divergence is allowed, $0 < p_{(l,n,m),(l',n',m')} < 1$ when $L_{l,n,m}$ and $L_{l',n',m'}$ are adjacent, while $p_{(l,n,m),(l',n',m')} = 0$ when $L_{l,n,m}$ and $L_{l',n',m'}$ are non-adjacent, and $\sum_{l' \in \mathcal{N}_{(l,n,m)}} p_{(l,n,m),l'} = 1$, where $\mathcal{N}_{(l,n,m)}$ is the set of all downstream neighboring links of $L_{l,n,m}$. The supply set $\boldsymbol{S}$ should be modified as

$$\boldsymbol{S} = \left\{ \begin{pmatrix} X_{l,n,m} \\ G_{n,m} \\ G_{l,I_{n,n'}} \end{pmatrix} : \begin{matrix} X_{1,2,1} \leq s_{2,1} G_{1,I_{2,2}} \\ X_{1,1,1} \leq s_{1,1} G_{1,I_{1,2}} \\ \vdots \\ X_{3,2,2} \leq s_{2,2} G_{2,I_{1,2}} \end{matrix} \right\} \tag{15}$$

where

$$X_{l,n,m} = \sum_{l',n',m'} X_{l',n',m'} p_{(l',n',m'),(l,n,m)} \tag{16}$$

Combining with (2), the demand set $\boldsymbol{D}$ can be rewritten by

$$\boldsymbol{D} = \left\{ (\boldsymbol{X}, T_s) : \begin{array}{c} \boldsymbol{P'X} \le \boldsymbol{S'}, \boldsymbol{X} \ge \boldsymbol{0} \\ \boldsymbol{1}^{\mathrm{T}}\boldsymbol{X} = T_s \end{array} \right\} \tag{17}$$

where $\boldsymbol{X} = \left[X_{1,1,1} \; X_{1,2,1} \; X_{1,1,2} \; X_{1,2,2} \right]^{\mathrm{T}}$, $\boldsymbol{1}$ is all one vector, T_s is the sum of demand, $\boldsymbol{S'}$ contains the maximum saturation flow at each intersection

$$\boldsymbol{S'} = \left[s_{2,1} \; s_{2,2} \; \max\left(s_{2,1}, s_{2,2}\right) \cdots \max\left(s_{1,1}, s_{1,2}\right) \right]^{\mathrm{T}}, \tag{18}$$

$\boldsymbol{P'}$ is the turning rate matrix that contains the sum of turning rate from each input flow to the links which intersect at an intersection, and thus the k-th row and n-th column element of $\boldsymbol{P'}$ can be expressed by

$$P'_{k,n} = \sum_{L_l \in I_k^-} \prod_{m=L_n \to L_l} p_{m,m+1} \tag{19}$$

where I_k^- is the set of incident links at the k-th intersection, L_n is the n-th input link, $L_n \to L_l$ represents the path from the n-th input link to the l-th incident link, $p_{m,m+1}$ is the turning rate from the m-th link to the $m+1$-th downstream neighboring link.

Remark 1: Note that the demand set given by (17) assumes that there is no flow generated or dissipated at each intersection or link. When considering the flow generation or dissipation, $\boldsymbol{S'}$ should be replaced by $\boldsymbol{S'} + \boldsymbol{W}$, where $\boldsymbol{W}$ is considered as an error term.

Recall that the maximum bottleneck delay at each exit of a link (intersection) is determined by the link flow, the saturation flow, the green-time proportion and the cycle duration of traffic light. We assume that the turning rates, i.e., the proportion of vehicles turn left, right or go straight etc., at each intersection can be estimated from empirical data. By taking into account the turning rates, (10) should be reformulated as

$$pX(1-G)\Delta T + pXb = psb \tag{20}$$

where the turning rate p appears both on the left and on the right side of (20), and thus it can be neglected. Then, by a given green-time proportion vector $\boldsymbol{G}$, the maximum bottleneck delays at signal 1 and 2 can be determined by

$$b_{n,m} = \frac{X_{3,n,m}\left(1 - G_{n,m}\right)\Delta T}{s_{n,m} - X_{3,n,m}} \tag{21}$$

The maximum bottleneck delay at each intersection can be expressed by

$$b_{l,I_{n,n'}} = \frac{X_{m,k,l}\left(1 - G_{l,I_{n,n'}}\right)\Delta T}{s_{k,l} - X_{m,k,l}} \tag{22}$$

where $k = n$ or n', $m = 1$, 2 or 3. Then, the equilibrium flow $X^*_{l,n,m}, l = 1, 2, 3; n, m = 1, 2$ can be determined by combining (8), (16), (21) and (22).

The equilibrium flow solved by (8) has a unique solution because $t_a(\boldsymbol{X}, \boldsymbol{G})$ is a non-decreasing function of $\boldsymbol{X}$ for a given $\boldsymbol{G}$. When no solution can be found within the set $\boldsymbol{S}$, that means, at least one route flow for each OD pair has approached the boundary of the set $\boldsymbol{S}$ for the given $\boldsymbol{G}$, then an ever-increasing queue will be generated on that route because more vehicles prefer to choose that cheaper route.

Note that $\boldsymbol{S}$, $\boldsymbol{D}$ and $\boldsymbol{F}$ given by (15), (17) and (1) can be extended to a general network with multiple OD pairs. We will show in the next subsection how this may be done.

B. Extension to a General Network With Multiple OD Pairs

Consider a general network with M OD pairs, each of which has N_m routes. Let us define $R_{n,m}$ as the n-th route of the m-th OD pair, $N_{I_{n,m}}$ as the number of intersections for the n-th route of the m-th OD pair, $X_{l,n,m}$ as the link flow at the l-th link of the n-th route of the m-th OD pair, $b_{l,n,m}$ as the maximum bottleneck delay at the l-th link of the n-th route of the m-th OD pair. Then, the number of intersections in this general network is

$$N_I = \sum_{m=1}^{M} \sum_{n=1}^{N_m} N_{I_{n,m}}$$

The demand set $\boldsymbol{D}$ of route flow vectors with non-negative components meeting all origin-destination demands is given by

$$\boldsymbol{D} = \left\{ (\boldsymbol{X}_{\mathrm{I}}, T_s) : \begin{array}{c} \boldsymbol{P'X_I} \le \boldsymbol{S'}, \boldsymbol{X}_{\mathrm{I}} \ge \boldsymbol{0} \\ \boldsymbol{1}^{\mathrm{T}}\boldsymbol{X}_{\mathrm{I}} = T_s \end{array} \right\} \tag{23}$$

where $\boldsymbol{X}_{\mathrm{I}}$ is the input flow vector with $\boldsymbol{X}_{\mathrm{I}} = \left[X_{1,1,1} \; X_{1,2,1} \cdots X_{1,N_M,M} \right]^{\mathrm{T}}$, $\boldsymbol{1}$ is an all-one vector, T_s is the sum of demands, $\boldsymbol{S'}$ contains the maximum saturated incident flow at N_I intersections and M exits of OD pairs with $\boldsymbol{S'} = \left[s_{1,\mathrm{m}} \; s_{2,\mathrm{m}} \cdots s_{N_I+M,\mathrm{m}} \right]^{\mathrm{T}}$, $\boldsymbol{P'}$ is the turning rate matrix given by (19). The feasible set $\boldsymbol{F}$ contains all feasible green-time proportion vectors, where the green time proportions at each intersection sum to one, are non-negative and given by

$$\boldsymbol{F} = \left\{ \begin{pmatrix} G_{n,m} \\ G_{m,I_{nm,n'm'}} \end{pmatrix} : \begin{array}{c} \sum_{n=1}^{N_m} G_{n,m} = 1 \\ \sum_{m,m'} G_{m,I_{nm,n'm'}} = 1 \\ G_{n,m} \ge 0 \\ G_{m,I_{nm,n'm'}} \ge 0 \\ m = 1 \cdots M \\ n = 1 \cdots N_m \end{array} \right\} \tag{24}$$

The supply set $\boldsymbol{S}$ contains link flow and green time proportion, for which each link flow is no greater than the saturation flow multiplied by the link green time proportion, and it is given by

$$\boldsymbol{S} = \left\{ \begin{pmatrix} X_{l,n,m} \\ G_{n,m} \\ G_{m,I_{nm,n'm'}} \end{pmatrix} : \begin{array}{c} X_{l,n,m} \le s_{n,m} G_{m,I_{nm,n'm'}} \\ X_{l,n,m} \ge 0 \\ m = 1 \cdots M \\ n = 1 \cdots N_m \end{array} \right\} \tag{25}$$

where the link flow $X_{l,n,m}$ is the sum of the flows at all incident links multiplied by the respective turning rate between the incident links and the current link.

By putting $X_{l,n,m}$ into the vector $\boldsymbol{X} = \left[X_{1,1,1} \; X_{2,1,1} \cdots X_{L_{N_M},N_M,M} \right]^{\mathrm{T}}$ and $b_{l,n,m}$ into the vector $\boldsymbol{b} = \left[b_{1,1,1}\ b_{2,1,1} \cdots b_{L_{N_M},N_M,M} \right]^{\mathrm{T}}$, the equilibrium flow on each link can be determined by finding a link-flow vector $\boldsymbol{X}^* = \left[X^*_{1,1,1}\ X^*_{2,1,1} \cdots X^*_{L_{N_M},N_M,M} \right]^{\mathrm{T}}$ and a green-time proportion vector $\boldsymbol{G}^* = \left[G^*_{1,1} \cdots G^*_{N_M,M}\ G^*_{1,I_{11,12}} \cdots G^*_{M,I_{N_M M,N_{M'} M'}} \right]^{\mathrm{T}}$ which minimize the sum of integrals of the link performance functions over the set of all $[\boldsymbol{X}, \boldsymbol{G}] \in \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$:

$$\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right] = \min_{[\boldsymbol{X},\boldsymbol{G}]} \sum_{m=1}^{M} \sum_{n=1}^{N_m} \sum_{l=1}^{L_{n,m}} \int_0^{X_{l,n,m}} t_{l,n,m}\left(\boldsymbol{X}, \boldsymbol{G}\right) dX \quad st. (23), (24), (25) \tag{26}$$

Remark 2: Note that $\boldsymbol{D} \times \boldsymbol{F}$ represents the Cartesian product of the sets $\boldsymbol{D}$ and $\boldsymbol{F}$. The symbol $\cap$ represents the intersection of two sets.

Note that when vertical queue is considered and bottleneck delay is neglected, (26) is equivalent to the weighted total travel time minimization problem [23], which aims at finding a solution minimizing the sum of product of link flow and link cost:

$$\begin{aligned}\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right] &= \min_{[\boldsymbol{X},\boldsymbol{G}]} \sum_{m=1}^{M} \sum_{n=1}^{N_m} \sum_{l=1}^{L_{n,m}} t_{l,n,m}\left(\boldsymbol{X}, \boldsymbol{G}\right) X_{l,n,m} \\ &= \min_{[\boldsymbol{X},\boldsymbol{G}]} \boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}, \boldsymbol{G}\right) \boldsymbol{X}\end{aligned} \tag{27}$$

The problem (27) is equivalent to finding a $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ such that $\left[-\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^*, \boldsymbol{G}^*\right), \boldsymbol{0}\right]$ is normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ (See [1] for the definition of "normal").

The reserve capacity can be determined by finding a $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ that maximizes the sum of input flow over the set of all $[\boldsymbol{X}, \boldsymbol{G}] \in \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$:

$$RC = \max_{\boldsymbol{X}^* \in \boldsymbol{\mathcal{X}}^*} \sum_{k=1}^{\sum_{m=1}^{M} N_m} X^*_{k,\mathrm{I}} - T_{\mathrm{c}} \tag{28}$$

where T_{c} is the current demand, $\boldsymbol{\mathcal{X}}^*$ is the set of equilibrium flow vectors corresponding to the given green-time proportion vectors in the feasible set $\boldsymbol{F}$:

$$\boldsymbol{\mathcal{X}}^* = \left\{ \begin{bmatrix} \boldsymbol{X}^* \\ \boldsymbol{G}^* \end{bmatrix} : \begin{matrix} \begin{bmatrix} \boldsymbol{X}^* \\ \boldsymbol{G}^* \end{bmatrix} = \min\limits_{[\boldsymbol{X},\boldsymbol{G}] \in \boldsymbol{D}_{T_{\mathrm{s}}} \times \boldsymbol{F} \cap \boldsymbol{S}} \boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}, \boldsymbol{G}\right) \boldsymbol{X} \\ \boldsymbol{D}_{T_{\mathrm{s}}} = \left\{ X_{\mathrm{I}} : \boldsymbol{1}^{\mathrm{T}} X_{\mathrm{I}} = T_{\mathrm{s}} \right\}, T_{\mathrm{s}} \in \boldsymbol{D} \end{matrix} \right\} \tag{29}$$

Note that the set of equilibrium flows and green-time proportion vectors given by (29) are obtained by searching $\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}, \boldsymbol{G}\right) \boldsymbol{X}$ over $\boldsymbol{S}$ and all $\boldsymbol{X}$ satisfying $\boldsymbol{1}^{\mathrm{T}} X_{\mathrm{I}} = T_{\mathrm{s}}$, where $T_s \in \boldsymbol{D}$, while (27) delivers a point by a given T_s. The problems (28) and (29) are equivalent to firstly determining $\boldsymbol{\mathcal{X}}^*$, which consists of $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ that lets $\left[-\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^*, \boldsymbol{G}^*\right), \boldsymbol{0}\right]$ be normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ to the set $\boldsymbol{D}_{T_{\mathrm{s}}} \times \boldsymbol{F} \cap \boldsymbol{S}$ for each $T_s \in \boldsymbol{D}$, and then finding the $\boldsymbol{X}^*$ such that $\boldsymbol{P} = \left[\boldsymbol{1}^{\mathrm{T}}_{\sum_{m=1}^{M} N_m}\ \boldsymbol{0}^{\mathrm{T}}_{N_L - \sum_{m=1}^{M} N_m} \right]^{\mathrm{T}}$ is normal at $\boldsymbol{X}^*$ to the set $\boldsymbol{\mathcal{X}}^*$, where $\boldsymbol{1}_K$ and $\boldsymbol{0}_K$ are $K \times 1$ all-one vector and all-zero vector, respectively.

Remark 3: Note that there is possibility that no $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ can be found for some $\boldsymbol{T_s} \in \boldsymbol{D}$ due to a lack of intersection between $\boldsymbol{D}_{T_{\mathrm{s}}}$ and $\boldsymbol{F} \cap \boldsymbol{S}$. It is also possible for some $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ that they approach a point at the boundary of $\boldsymbol{D}_{T_{\mathrm{s}}} \times \boldsymbol{F} \cap \boldsymbol{S}$, at which $\left[-\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^*, \boldsymbol{G}^*\right), \boldsymbol{0}\right]$ is not normal to the supporting hyperplane. In these cases, $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ are removed from $\boldsymbol{\mathcal{X}}^*$.

C. Applying the P0 Control Policy and a Novel Control Policy to the General Network to Maximize the Network Capacity With a Steady Demand and Vertical Queue

Directly solving the equilibrium flow and green-time proportion vector from (27) requires knowing the sum of input flow T_{s}, which is hard to realize in practice. Smith has proved that with the P0 control policy and adjustable bottleneck delay, the capacity can be maximized in a simple network by re-routing and traffic signal assignment without knowing the demand information. Prior to applying the P0 control policy to the proposed general network, we firstly introduce the original P0 control policy proposed by Smith and then give the condition, under which the P0 control policy can be utilized to maximize the network capacity with a steady demand and the network reserve capacity.

Consider a simple network with just two routes, the original P0 control policy aims at equalizing the product of saturation flow and bottleneck delay by selecting appropriate green-time proportion vector, that is, $s_1 b_1 = s_2 b_2$ [1]. If this is not possible, then the policy should select green-time proportion vector to guarantee two numbers to be as equal as possible. For the proposed general network with multiple OD pairs, the P0 control policy aims at equalizing the stage pressure $P_{n,m} = \sum_{l=1}^{N_{n,m}} s_{l,n,m} b_{l,n,m} = \sum_{l=1}^{N_{n,m}} P_{l,n,m}, n = 1, 2 \cdots N_m$ for M OD pairs by selecting appropriate green-time proportion vectors. The stage pressure $P_{n,m}$ is defined by adding the terms $s_{l,n,m} b_{l,n,m}$ over the links on the n-th route of the m-th OD pair and $P_{l,n,m}$ is the pressure at the (l, n, m)-th link.

Because the target of the P0 control policy may not always be achieved or even achievable although the green-time proportion vector can be slowly adjusted to approach the target [1], the P0 control policy for the simple network in [1] was stated as

> *For any $\boldsymbol{X}$ and delay vector $\boldsymbol{b}$, choose a stage green-time proportion vector $\boldsymbol{G}$ in $\boldsymbol{F}$ so that the stage pressure vector $\boldsymbol{P}(\boldsymbol{b})$ is normal at $\boldsymbol{G}$ to $\boldsymbol{F}$.*

Recall that in our proposed general network, there are $N_{I_{n,m}}$ traffic signals at all intersections for the n-th route of the m-th OD pair and one traffic signal at the merging point, and thus there are $N_{I_{n,m}} + 1$ adjustable green-time proportions for the n-th route of the m-th OD pair. Note that any changes of the green-time proportions at the $N_{I_{n,m}}$ intersections will influence the equilibrium flow for other OD pairs because of $\boldsymbol{F}$ given by (24). Recall that we assume that the cycle duration ΔT for each traffic signal is same, thus, the P0 control policy for the general network can be restated as

> *For any $\boldsymbol{X}$ and delay vector $\boldsymbol{b}$ choose a stage green-time proportion vector $\boldsymbol{G} = \left[G_{1,1,1} \cdots G_{l,n,m} \cdots G_{L_{N_M},N_M,M} \right]^T$ in $\boldsymbol{F}$ to maximize $\sum_{m=1}^{M} \sum_{n=1}^{N_m} \sum_{l=1}^{N_n} s_{l,n,m} b_{l,n,m} G_{l,n,m}$, that is, to let $\boldsymbol{P}(\boldsymbol{b}) = \left[P_{1,1,1} \cdots P_{l,n,m} \cdots P_{L_{N_M},N_M,M} \right]^T$ be normal at $\boldsymbol{G}$ to $\boldsymbol{F}$.*

Note that here we represent the green-time proportion by $G_{l,n,m}$ instead of $G_{l,I_{n,n'}}$ occurring in $\boldsymbol{F}$ and $\boldsymbol{S}$ for the sake of simplicity of expression, $G_{l,n,m}$ stands for the green-time proportion assigned to the (l, n, m)-th link.

In what follows, we will give the condition that P0 control policy can maximize the general network capacity with vertical queue. Recall that the problem (27) is to find a $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ such that $\left[-\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right), \boldsymbol{0}\right]$ is normal at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, the equilibrium travel cost $\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right)$ varies with $\boldsymbol{X}^{*}$ and $\boldsymbol{G}^{*}$ because it consists of the free travel cost $\boldsymbol{C}$ (constant) and the bottleneck delay $\boldsymbol{b}$ (non-negative function of $\boldsymbol{X}^{*}$ and $\boldsymbol{G}^{*}$). Suppose that there is a $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ at which $[-\boldsymbol{C}, \boldsymbol{0}]$ is normal to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$. Then a question arises: what is the condition that lets $\left[-\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right) \ \boldsymbol{P}(\boldsymbol{b})\right]$ be normal at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, where $\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right)=\boldsymbol{C}+\boldsymbol{b}(\boldsymbol{X}^{*}, \boldsymbol{G}^{*})$?

The supply set $\boldsymbol{S}$ is defined by a set of linear inequalities and thus it is the intersection of a finite number of closed half-spaces. The demand set $\boldsymbol{D}$ and the feasible set $\boldsymbol{F}$ are subset of an affine space. The three sets are compact and the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ is the intersection of $\boldsymbol{D}$ and $\boldsymbol{S}$ by given $\boldsymbol{F}$ and thus a subset of an affine space. Now suppose that $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S} \neq \emptyset$ and let the half spaces of $\boldsymbol{S}$ be $\boldsymbol{H}_{s1}\ \boldsymbol{H}_{s2} \cdots \boldsymbol{H}_{sL}$, where L is the number of links for the M OD pairs. Each half-space $\boldsymbol{H}_{sl}$ can be expressed by $\boldsymbol{H}_{sl}=\{(X, G): X_{l} \leq s_{l} G_{l}\}$, where $\boldsymbol{G}_{l}$ is the green-time proportion assigned to the l-th link. The boundaries of $\boldsymbol{S}$ is perpendicular to each other, thus we can define a $2L \times 1$ vector $\boldsymbol{n}_{sl}=\left[\boldsymbol{0}^{\mathrm{T}}\ 1\ \boldsymbol{0}^{\mathrm{T}}\ -s_{l}\ \boldsymbol{0}^{\mathrm{T}}\right]^{\mathrm{T}}$, which is non-zero and normal to the half-space $\boldsymbol{H}_{sl}$. The first $\boldsymbol{0}^{\mathrm{T}}$ is a $1 \times l-1$ zero vector, the second $\boldsymbol{0}^{\mathrm{T}}$is a $1 \times L-1$ zero vector and the third $\boldsymbol{0}^{\mathbf{T}}$ is a $1 \times L-l$ zero vector. The vector $\boldsymbol{n}_{sl}$ is perpendicular to $\boldsymbol{n}_{sl'}$ for $l \neq l'$. We define $\boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}$ as a non-zero vector normal to the set $\boldsymbol{D} \times \boldsymbol{F}$. Recall that $\boldsymbol{D} \times \boldsymbol{F}$ is a subset of an affine space, thus $\boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}$ is normal to $\boldsymbol{D} \times \boldsymbol{F}$ at any elements at its boundary.

Because we assume that $[-\boldsymbol{C}, \boldsymbol{0}]$ is normal at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, then $[-\boldsymbol{C}, \boldsymbol{0}]$ can be expressed by a non-negative combination of $\boldsymbol{n}_{sl}$ and $\boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}$, that is, $[-\boldsymbol{C}, \boldsymbol{0}]=\sum_{l=1}^{L} w_{sl} \boldsymbol{n}_{sl}+w_{D\times F} \boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}$, where w_{sl} and $w_{D\times F}$ are non-negative weights. From Definition 1, the P0 control policy can maximize the general network capacity at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ which means that $[-(\boldsymbol{C}+\boldsymbol{b}), \boldsymbol{P}(\boldsymbol{b})]$ is normal at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ to $\boldsymbol{D} \times \boldsymbol{F}$, i.e., $-(\boldsymbol{C}+\boldsymbol{b})$ is normal at $\boldsymbol{X}^{*}$ to $\boldsymbol{D}$ and $\boldsymbol{P}(\boldsymbol{b})$ is normal at $\boldsymbol{G}^{*}$ to $\boldsymbol{F}$. If $[-\boldsymbol{C}, \boldsymbol{0}]$ is perpendicular to the supporting hyperplane of the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$, then $[-\boldsymbol{C}, \boldsymbol{0}]$ is scaled by $\boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}$ and thus $w_{sl}=0$ for $l=1 \cdots L$. Thus, the condition for the general network capacity maximization is that $\boldsymbol{H}\ \boldsymbol{b}^{*}=k_{1} \boldsymbol{H} \boldsymbol{C}$ and $\boldsymbol{G}^{*}=k_{2} \boldsymbol{s} \bullet \boldsymbol{b}^{*}$, where k_{1} and k_{2} are the scaling factors, $\boldsymbol{H}$ is an $\sum_{m=1}^{M} N_{m} \times L$ matrix with $\left[\ \mathbf{0}_{(r-1) L_{r-1}}\ \mathbf{1}_{L_{r}}\ \mathbf{0}\ \right]$ on its r-th row, $\boldsymbol{s} \bullet \boldsymbol{b}^{*}$ represents the point-wise multiplication between $\boldsymbol{s}$ and $\boldsymbol{b}^{*}$.

When $[-\boldsymbol{C}, \boldsymbol{0}]$ is not perpendicular to the supporting hyperplane of the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ at $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$, $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ must be located at the boundary of $\boldsymbol{S}$ and $\boldsymbol{D}$ because the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ is a subset of an affine space and $-\boldsymbol{C}$ is a constant vector. Then $w_{sl} \neq 0$ and $w_{D\times F} \boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}=[-\boldsymbol{C}, \boldsymbol{0}]-\sum_{l=1}^{L} w_{sl} \boldsymbol{n}_{sl}$. Let b_{l}^{*} be w_{sl}, $\sum_{l=1}^{L} w_{sl} \boldsymbol{n}_{sl}$ becomes $\left[\ \boldsymbol{b}^{*\mathrm{T}}\ -\boldsymbol{s}^{\mathrm{T}} \bullet \boldsymbol{b}^{*\mathrm{T}}\ \right]^{\mathrm{T}}$ and thus the condition for the general network capacity maximization is that $\left[\ -\left(\boldsymbol{C}+\boldsymbol{b}^{*}\right)^{\mathrm{T}}\ \boldsymbol{s}^{\mathrm{T}} \bullet \boldsymbol{b}^{*\mathrm{T}}\ \right]^{\mathrm{T}}=w_{D\times F} \boldsymbol{n}_{\boldsymbol{D}\times\boldsymbol{F}}$. Note that $\boldsymbol{b}^{*}$ is not normal to $\boldsymbol{S}$ if $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$ is located at the interior of the half-spaces $\boldsymbol{H}_{sl}$ but b_{l}^{*} is not zero.

The two conditions indicate that by proper construction of $\boldsymbol{b}^{*}$, the P0 control policy can maximize the general network capacity. Recall that the bottleneck delay is a non-linear function of equilibrium link flow and green-time proportion vector (11), $\boldsymbol{b}^{*}$ can only be constructed by adjusting the green-time proportion vector. Then, we can obtain the following lemma:

Lemma 1: If the proposed two conditions are not fulfilled, no $\boldsymbol{b}^{*}$ can be found through adjusting the green-time proportion vector to maximize capacity of the general network using P0 control policy. By adjusting the green-time proportion vector, the equilibrium flow $\boldsymbol{X}^{*}$ can be maintained as long as $-\left[\boldsymbol{C}+\boldsymbol{b}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}+\boldsymbol{\Delta G}\right)\right]^{T}=w_{D} \boldsymbol{n}_{\boldsymbol{D}}$ and $\boldsymbol{X}^{*}$ is within the set $\boldsymbol{S}$ for the new green-time vector $\boldsymbol{G}^{*}+\boldsymbol{\Delta G}$.

Proof: The normality of the travel cost vector $-(\boldsymbol{C}+\boldsymbol{b})^{\mathrm{T}}$ to the demand set $\boldsymbol{D}$ can be ensured by adjusting green-time proportion vector to change $\boldsymbol{b}$, which is however unable to guarantee that $\boldsymbol{s} \bullet \boldsymbol{b}$ is normal to the feasible set $\boldsymbol{F}$. That means, it is possible to find another $\boldsymbol{G}_{\mathrm{m}}$ in $\boldsymbol{F}$ that lets $(\boldsymbol{s} \bullet \boldsymbol{b})^{\mathbf{T}} \boldsymbol{G}_{\mathrm{m}}$ be larger than $(\boldsymbol{s} \bullet \boldsymbol{b})^{\mathbf{T}}\left(\boldsymbol{G}^{*}+\boldsymbol{\Delta G}\right)$ and thus $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}+\boldsymbol{\Delta G}\right]$ may not be the solution to maximize capacity of the general network using P0 control policy. From the definition of $\boldsymbol{D}$, $\boldsymbol{S}$ and capacity maximization, it is easy to get the condition to keep the equilibrium flow $\boldsymbol{X}^{*}$ unchanged by adjusting the green-time proportion vector. □

Lemma 1 suggests that network capacity cannot always be maximized with P0 control policy by adjusting green-time proportion vector. The prerequisite of capacity maximization is the capability to freely construct bottleneck delay $\boldsymbol{b}$ such as changing the length of queue at each link exit, which is consistent with Smith's statement. To relieve the condition of capacity maximization by adjusting green-time proportion vector for any $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$, at which $[-\boldsymbol{C}, \boldsymbol{0}]$ is normal to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, we propose a novel control policy in this paper, which can be stated as

> *For any $\boldsymbol{X}$ and delay vector $\boldsymbol{b}$, choose a stage green-time proportion vector $\boldsymbol{G}$ in $\boldsymbol{F}$ to maximize $\sum_{m=1}^{M} \sum_{n=1}^{N_{m}} \sum_{l=1}^{N_{n}} k_{l,n,m}\left(C_{l,n,m}+b_{l,n,m}\right) G_{l,n,m}$, that is, to let $\boldsymbol{P}(\boldsymbol{b})=\boldsymbol{k} \bullet(\boldsymbol{C}+\boldsymbol{b})$ be normal at $\boldsymbol{G}$ to $\boldsymbol{F}$, where $\boldsymbol{k}$ is the scaling vector between the normal vector $\boldsymbol{n}_{\boldsymbol{D}}$ to $\boldsymbol{D}$ at $\boldsymbol{X}$ and the normal vector $\boldsymbol{n}_{\boldsymbol{F}}$ to $\boldsymbol{F}$ at $\boldsymbol{G}$, i.e., $\boldsymbol{n}_{\boldsymbol{F}}=\boldsymbol{k} \bullet \boldsymbol{n}_{\boldsymbol{D}}$.*

Because $\boldsymbol{D}$ and $\boldsymbol{F}$ are subset of affine spaces by the definition, $\boldsymbol{n}_{\boldsymbol{D}}$ and $\boldsymbol{n}_{\boldsymbol{F}}$ are constant for any $\boldsymbol{X}$ and $\boldsymbol{G}$ on $\boldsymbol{D}$ and $\boldsymbol{F}$, respectively. Then, $\boldsymbol{k}$ is also constant. The following lemma can be obtained:

Lemma 2: For any solution $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}\right]$, at which $[-\boldsymbol{C}, \boldsymbol{0}]$ is normal to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, a bottleneck delay $\boldsymbol{b}^{*}=\boldsymbol{b}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}+\boldsymbol{\Delta G}\right)$ can be found by adjusting the green-time proportion vector $\boldsymbol{G}$ in $\boldsymbol{F}$ so that the solution $\left[\boldsymbol{X}^{*}, \boldsymbol{G}^{*}+\boldsymbol{\Delta G}, \boldsymbol{b}^{*}\right]$ is consistent with the novel control policy if $\boldsymbol{X}^{*}$ is within set $\boldsymbol{S}$ for the new green-time vector $\boldsymbol{G}^{*}+\boldsymbol{\Delta G}$ and $-\left[\boldsymbol{C}+\boldsymbol{b}\left(\boldsymbol{X}^{*}, \boldsymbol{G}^{*}+\boldsymbol{\Delta G}\right)\right]^{T}=w_{D} \boldsymbol{n}_{\boldsymbol{D}}$.

Proof: Let us define a $2L \times 1$ normal vector to the half spaces of $\boldsymbol{S}$: $\boldsymbol{H}_{s1}\ \boldsymbol{H}_{s2} \cdots \boldsymbol{H}_{sL}$ by

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

$\boldsymbol{n_{sl}} = \left[\boldsymbol{0}^{\mathrm{T}}\ \boldsymbol{b_l}\ \boldsymbol{0}^{\mathrm{T}}\ -\boldsymbol{k_l}\left(\boldsymbol{C}+\boldsymbol{b_l}\right)\ \boldsymbol{0}^{\mathrm{T}}\right]^{\mathrm{T}}$. By the given condition, $[-\boldsymbol{C}, \boldsymbol{0}]$ is normal to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ at $\left[\boldsymbol{X}^*, \boldsymbol{G}^* + \boldsymbol{\Delta G}\right]$ for any $\boldsymbol{\Delta G}$, then $[-\boldsymbol{C}, \boldsymbol{0}] = \sum_{l=1}^{L} w_{sl}\boldsymbol{n_{sl}} + w_{D\times F}\boldsymbol{n_{D\times F}}$, where $\boldsymbol{n_{D\times F}}$ must be normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^* + \boldsymbol{\Delta G}\right]$ to $\boldsymbol{D} \times \boldsymbol{F}$ and $\sum_{l=1}^{L} w_{sl}\boldsymbol{n_{sl}}$ must be normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^* + \boldsymbol{\Delta G}\right]$ to $\boldsymbol{S}$. Then, we have $w_{D\times F}\boldsymbol{n_{D\times F}} = [-\boldsymbol{C}, \boldsymbol{0}] - \sum_{l=1}^{L} w_{sl}\boldsymbol{n_{sl}}$. Let w_{sl} be one, then $w_{D\times F}\boldsymbol{n_{D\times F}} = \begin{bmatrix} -\left(\boldsymbol{C}+\boldsymbol{b}\left(\boldsymbol{X}^*, \boldsymbol{G}^*+\boldsymbol{\Delta G}\right)\right)^{\mathrm{T}}, \\ \boldsymbol{k}^{\mathrm{T}} \bullet \left(\boldsymbol{C}+\boldsymbol{b}\left(\boldsymbol{X}^*, \boldsymbol{G}^*+\boldsymbol{\Delta G}\right)\right)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$. To guarantee that $\boldsymbol{n_{D\times F}}$ is normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^* + \boldsymbol{\Delta G}\right]$ to $\boldsymbol{D} \times \boldsymbol{F}$, $-\left(\boldsymbol{C}+\boldsymbol{b}\left(\boldsymbol{X}^*, \boldsymbol{G}^*+\boldsymbol{\Delta G}\right)\right)^{\mathrm{T}}$ should be normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^* + \boldsymbol{\Delta G}\right]$ to $\boldsymbol{D}$, and $\boldsymbol{k}^{\mathrm{T}} \bullet \left(\boldsymbol{C}+\boldsymbol{b}\left(\boldsymbol{X}^*, \boldsymbol{G}^*+\boldsymbol{\Delta G}\right)\right)^{\mathrm{T}}$ should be normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^* + \boldsymbol{\Delta G}\right]$ to $\boldsymbol{F}$. Because of $\boldsymbol{n_F} = \boldsymbol{k} \bullet \boldsymbol{n_D}$ and the condition for flow maintenance proposed by Lemma 1, Lemma 2 is proved. □

Lemma 2 indicates that any bottleneck delay $\boldsymbol{b}$, with which $-(\boldsymbol{C}+\boldsymbol{b})^{\mathrm{T}}$ is normal to $\boldsymbol{D}$, fulfills that $\boldsymbol{P}(\boldsymbol{b})$ is also normal to $\boldsymbol{F}$.

Now we have shown the superiority of the novel control policy over the P0 control policy because it relieves the condition for capacity maximization. Note that Lemma 2 states a sufficient condition for determining the equilibrium by adjusting the green-time proportion vector and fixing the equilibrium flow vector with the novel control policy. In next subsection, we propose a dynamical system which combines flow swap process, green-time proportion swap process and flow divergence The proof of the existence, uniqueness and stability of equilibrium will be given.

IV. A Hybrid Dynamical System Incorporating Flow Swap Process, Green-Time Proportion Swap Process and Flow Divergence

In this subsection, we firstly describe a dynamical system for the proposed general network and then give the proof of equilibrium existence, condition of uniqueness and stability.

A. Hybrid Dynamical System for the General Network

In this study, we apply the proportional switch adjustment process introduced by Smith to our proposed general network. The proportional switch adjustment process (PAP) can be utilized to change flow vector and green-time proportional vector at each time instant. For the case where there are several routes joining a single OD pair [1], the flow change on each route at each time instant can be expressed by

$$\boldsymbol{\Delta X_{\mathrm{SP}}} = \sum_{\{(r:s):r<s\}} w \left\{ \begin{array}{l} X_r\left[C_r(\boldsymbol{X}) - C_s(\boldsymbol{X})\right]_+ \boldsymbol{\Delta_{rs}} + \\ X_s\left[C_s(\boldsymbol{X}) - C_r(\boldsymbol{X})\right]_+ \boldsymbol{\Delta_{sr}} \end{array} \right\} \tag{30}$$

where r and s represent the r-th and s-th routes for the OD pair, respectively, X_r and X_s represent the flow on the r-th and s-th route at the current time instant, respectively, $\boldsymbol{X}$ is the route flow vector, $C_r(\boldsymbol{X})$ and $C_s(\boldsymbol{X})$ represent the travel cost on the r-th and s-th routes with the flow vector $\boldsymbol{X}$, respectively, $\left[C_r(\boldsymbol{X}) - C_s(\boldsymbol{X})\right]_+ = \max\left\{\left[C_r(\boldsymbol{X}) - C_s(\boldsymbol{X})\right], 0\right\}$, $\boldsymbol{\Delta_{rs}} = \left[\boldsymbol{0}^{\mathrm{T}}\ -1\ \boldsymbol{0}^{\mathrm{T}}\ 1\ \boldsymbol{0}^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{\Delta_{sr}} = \left[\boldsymbol{0}^{\mathrm{T}}\ 1\ \boldsymbol{0}^{\mathrm{T}}\ -1\ \boldsymbol{0}^{\mathrm{T}}\right]^{\mathrm{T}}$, w is a positive scaling factor. Note that $\boldsymbol{\Delta_{rs}}$ and $\boldsymbol{\Delta_{sr}}$ are $N_m \times 1$ vectors, the first, second and third $\boldsymbol{0}^{\mathrm{T}}$ are $r-1\times 1$ vector, $s-r-1\times 1$ vector and $N_m - s \times 1$ vector, respectively. Equation (30) states that at each time instant, the flow swapping from a higher cost route to a lower cost route is proportional to the product of flow on the higher cost route and the cost difference between the two routes, the flow swap vector $\boldsymbol{\Delta X_{\mathrm{SP}}}$ is determined by summing the flow swaps between any pair of routes.

In our proposed general network, link flow on the links along the same route is different because of flow diverge at each link exit. The flow swaps between routes denote the input flow re-assignment between any pair of routes for each OD pair. Recall that the Wardrop equilibrium $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ can be determined by finding a $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ such that $\left[-\boldsymbol{t}^{\mathrm{T}}\left(\boldsymbol{X}^*, \boldsymbol{G}^*\right), \boldsymbol{0}\right]$ is normal at $\left[\boldsymbol{X}^*, \boldsymbol{G}^*\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ (27). In the proposed general network, the $L \times 1$ link flow vector $\boldsymbol{X}$ has the degree of freedom of the number of input links for the M OD pairs, i.e., $\sum_{m=1}^{M} N_m$, and $L - \sum_{m=1}^{M} N_m$ elements in $\boldsymbol{X}$ can be determined by the input link flow vector $\boldsymbol{X}_{\mathrm{I}}$ and the turning rates between two adjacent links. Therefore, (27) can be rewritten by

$$\begin{aligned} \left[\boldsymbol{X}_{\mathrm{I}}^*, \boldsymbol{G}^*\right] &= \min_{[\boldsymbol{X},\boldsymbol{G}]} \boldsymbol{t}^{\mathrm{T}}(\boldsymbol{X}, \boldsymbol{G})\,\boldsymbol{X} \\ &= \min_{[\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]} \boldsymbol{t}_{\mathrm{I}}^{\mathrm{T}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G})\boldsymbol{X}_{\mathrm{I}} \end{aligned} \tag{31}$$

where $\boldsymbol{t}_{\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G})$ is the $\sum_{m=1}^{M} N_m$ dimensional travel cost vector projected from $\boldsymbol{t}(\boldsymbol{X}, \boldsymbol{G})$ and can be expressed by

$$\begin{aligned} \boldsymbol{t}_{\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) &= \boldsymbol{P'}^{\mathrm{T}}\boldsymbol{t}(\boldsymbol{X}, \boldsymbol{G}) \\ &= \boldsymbol{P'}^{\mathrm{T}}(\boldsymbol{C} + \boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G})) \\ &= \boldsymbol{P'}^{\mathrm{T}}\boldsymbol{C} + \boldsymbol{P'}^{\mathrm{T}}\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) \\ &= \boldsymbol{C}_{\mathrm{I}} + \boldsymbol{b}_{\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) \end{aligned} \tag{32}$$

where $\boldsymbol{P'}$ is the turning rate matrix given by (19), $\boldsymbol{C}_{\mathrm{I}}$ and $\boldsymbol{b}_{\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G})$ are the projected free-travel cost vector and bottleneck delay vector, respectively. Then, $\boldsymbol{X} = \boldsymbol{P'}\boldsymbol{X}_{\mathrm{I}}$ and thus problem (27) becomes finding a $\left[\boldsymbol{X}_{\mathrm{I}}^*, \boldsymbol{G}^*\right]$ such that $\left[-\boldsymbol{t}_{\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}}^*, \boldsymbol{G}^*\right), \boldsymbol{0}\right]$ is normal at $\left[\boldsymbol{X}_{\mathrm{I}}^*, \boldsymbol{G}^*\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$. Thus, the input flow swap vector for M OD pairs $\boldsymbol{\Delta X}_{m,\mathrm{I}} = \left[\varDelta X_{1,1,m}\ \varDelta X_{1,2,m}\ \cdots\ \varDelta X_{1,N_m,m}\right]^{\mathrm{T}}$ can be determined by

$$\begin{aligned} \boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}} &= \sum_{\{(r:s):r<s\}} w \left\{ \begin{array}{c} X_{1,r,m}\begin{bmatrix} t_{r,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) - \\ t_{s,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) \end{bmatrix}_+ \\ \times \boldsymbol{\Delta_{rs}} + \boldsymbol{\Delta_{sr}} \times \\ X_{1,s,m}\begin{bmatrix} t_{s,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) - \\ t_{r,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) \end{bmatrix}_+ \end{array} \right\} \\ &= \sum_{\{(r:s):r<s\}} w \left\{ \begin{array}{c} X_{1,r,m}\begin{bmatrix} C_{r,m,\mathrm{I}} - C_{s,m,\mathrm{I}} + \\ b_{r,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) - \\ b_{s,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) \end{bmatrix}_+ \\ \times \boldsymbol{\Delta_{rs}} + \boldsymbol{\Delta_{sr}} \times \\ X_{1,s,m}\begin{bmatrix} C_{s,m,\mathrm{I}} - C_{r,m,\mathrm{I}} + \\ b_{s,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) - \\ b_{r,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}}, \boldsymbol{G}) \end{bmatrix}_+ \end{array} \right\} \end{aligned} \tag{33}$$

where $X_{1,r,m}$ and $X_{1,s,m}$ are respectively the input flow on the r-th and s-th route for the m-th OD pair, $C_{r,m,\mathrm{I}}$, $C_{s,m,\mathrm{I}}$, $b_{r,m,\mathrm{I}}$ and $b_{s,m,\mathrm{I}}$ are respectively the free-travel travel costs and bottleneck delay on the r-th and s-th input links for the m-th

OD pair. Then the link flow swap vector for the m-th user can be determined by

$$\boldsymbol{\Delta X_m} = \boldsymbol{A_m P'} \begin{bmatrix} \boldsymbol{\Delta X_{1,\mathrm{I}}} \\ \boldsymbol{\Delta X_{2,\mathrm{I}}} \\ \vdots \\ \boldsymbol{\Delta X_{M,\mathrm{I}}} \end{bmatrix} \tag{34}$$

where $\boldsymbol{A_m}$ is the m-th scaling matrix.

To give the green-time proportion adjustment process in the general network, we firstly give $\boldsymbol{\Delta G_{\mathrm{SP}}}$ for the single OD pair case [1]:

$$\boldsymbol{\Delta G}_{\mathrm{SP}} = \sum_{\{(r:s):r<s\}} w \left\{ \begin{array}{c} G_r \begin{bmatrix} [sb\,(\boldsymbol{X},\boldsymbol{G})]_s - \\ [sb\,(\boldsymbol{X},\boldsymbol{G})]_r \end{bmatrix}_+ \\ \times \boldsymbol{\Delta_{rs}} + \boldsymbol{\Delta_{sr}} \times \\ G_s \begin{bmatrix} [sb\,(\boldsymbol{X},\boldsymbol{G})]_r - \\ [sb\,(\boldsymbol{X},\boldsymbol{G})]_s \end{bmatrix}_+ \end{array} \right\} \tag{35}$$

where G_r and G_s are green-time proportion vectors on the r-th and s-th route, respectively, $[sb\,(\boldsymbol{X},\boldsymbol{G})]_s$ and $[sb\,(\boldsymbol{X},\boldsymbol{G})]_r$ are stage pressure on the s-th and r-th routes, respectively. Equation (35) shows that by P0 control policy, green-time proportion swaps from a lower stage-pressure route to a higher stage-pressure route, and the green-time proportion swap vector is proportional to the product of green-time proportion on the lower stage-pressure route and stage pressure difference between the two routes, $\boldsymbol{\Delta G}_{\mathrm{SP}}$ is determined by summing the green-time proportion swaps between any pair of routes.

Note that Smith only considered that one signal per route that can be adjusted [1]. In our proposed general network, each route of an OD pair traverses multiple traffic signals, and thus adjustment of each signal can affect equilibrium flow and bottleneck delay on that route. It is hard to directly give the green-time proportion swap vector for each route of each OD pair by (35) because the red-green splits of any signals at intersections have impacts on the routes for different OD pairs. Recall that P0 control policy aims at equalizing the stage pressure $P_{n,m} = \sum_{l=1}^{N_{n,m}} s_{l,n,m} b_{l,n,m}$ on each route, i.e., maximizing $\sum_{m=1}^{M}\sum_{n=1}^{N_m}\sum_{l=1}^{N_n} s_{l,n,m} b_{l,n,m} G_{l,n,m}$. From the feasible set defined by (24), adjustment of green-time proportions of traffic signal at different intersections is independent of each other while the green-time proportions for the links incident to the same intersection must sum to one. Thus, the green-time proportion adjustment process in the general network can be realized by simultaneously adjusting the green-time proportion vector for traffic signal at each intersection so that the stage pressure is normal at the adjusted green-time proportion vector to the set $\sum_{m,m'} G_{m,I_{nm,n'm'}} = 1$ for traffic signal at each intersection:

$$\boldsymbol{\Delta G}_{i,\mathrm{GP}} = \sum_{\{(r:s):r<s\}} w \left\{ \begin{array}{c} G_{r,i} \begin{bmatrix} [P\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))]_{s,i} - \\ [P\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))]_{r,i} \end{bmatrix}_+ \\ \times \boldsymbol{\Delta_{rs}} + \boldsymbol{\Delta_{sr}} \times \\ G_{s,i} \begin{bmatrix} [P\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))]_{r,i} - \\ [P\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))]_{s,i} \end{bmatrix}_+ \end{array} \right\} \tag{36}$$

where $i = 1, 2, \cdots I$, r and s are the two links incident to the i-th intersection and they may be affiliated with different OD pairs, $[P\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))]_{s,i}$ and $[P\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))]_{r,i}$ are respectively the stage pressure on the s-th and r-th links incident to the i-th intersection, I is the number of intersections in the general network. With the novel control policy proposed in subsection III-C, the stage pressure vector $\boldsymbol{P}\,(\boldsymbol{b}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))$ should be changed to $\boldsymbol{k} \bullet (\boldsymbol{C} + \boldsymbol{b}\,(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}))$, the rest are the same as those with P0 control policy.

Let us define $\boldsymbol{\Delta X_m}$ and $\boldsymbol{\Delta G_m}$ as the link flow swap vector and green-time proportion swap vector for the m-th OD pair obtained by (33), (34) and (36), respectively. Then, the dynamical system can be obtained by

$$\begin{aligned} [\boldsymbol{X_m},\boldsymbol{G_m}](\boldsymbol{t}+\boldsymbol{1}) &= [\boldsymbol{X_m},\boldsymbol{G_m}](\boldsymbol{t}) + \lambda\,(t)\,\boldsymbol{\Delta}[\boldsymbol{X_m},\boldsymbol{G_m}](\boldsymbol{t}) \\ m &= 1 \cdots M \end{aligned} \tag{37}$$

where $\lambda\,(t)$ is a positive number that represents the step length at the t-th time instant and we suppose that it follows the step length rules proposed in [25].

B. The Condition for Existence, Uniqueness and Stability of Equilibrium of the Dynamical System Given by (37)

Let us define the boundary of the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ as $\boldsymbol{B}$ and express the projected travel cost vector $\boldsymbol{t}_{\mathrm{I}}\,(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})$ by $t_{n_m,m,\mathrm{I}}\,(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})$ for $n_m = 1, 2, \cdots, N_m; m = 1, 2, \cdots, M$. Then we can obtain the following lemma and corollary.

Lemma 3: The Wardrop equilibrium of the dynamical system given by (37) exists when $t_{n_m,m,I}\,(\boldsymbol{X_I},\boldsymbol{G})$ and $P\,(b\,(\boldsymbol{X_I},\boldsymbol{G}))$ are continuous functions over the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ for $n_m = 1, 2, \cdots, N_m; m = 1, 2, \cdots, M$, and for any element $\boldsymbol{Y} = [\boldsymbol{X_I},\boldsymbol{G}]$ on $\boldsymbol{B}$, it satisfies the condition that $A_{ux,\boldsymbol{m}}\boldsymbol{\Delta X_{m,I}} \leq \boldsymbol{0}, \boldsymbol{m} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{M}$ and $A_{ug,\boldsymbol{m}}\boldsymbol{\Delta G_m} \leq \boldsymbol{0}, \boldsymbol{m} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{M}$ for all elements in $\boldsymbol{Y}$ reaching their upper boundaries; while $A_{lx,\boldsymbol{m}}\boldsymbol{\Delta X_{m,I}} \geq \boldsymbol{0}, \boldsymbol{m} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{M}$ and $A_{lg,\boldsymbol{m}}\boldsymbol{\Delta G_m} \geq \boldsymbol{0}, \boldsymbol{m} = \boldsymbol{1}, \boldsymbol{2}, \cdots, \boldsymbol{M}$ for all elements in $\boldsymbol{Y}$ reaching their lower boundaries, where $A_{ux,\boldsymbol{m}}$, $A_{ug,\boldsymbol{m}}$, $A_{lx,\boldsymbol{m}}$ and $A_{lg,\boldsymbol{m}}$ are the scaling matrices for the elements in $\boldsymbol{Y}$ reaching their upper and lower boundaries, respectively.

Proof: Let us rewrite (37) as

$$\begin{aligned} [\boldsymbol{X_m},\boldsymbol{G_m}](\boldsymbol{t}+\boldsymbol{1}) &= \boldsymbol{f}\,([\boldsymbol{X_m},\boldsymbol{G_m}]\,(\boldsymbol{t})\,,\boldsymbol{\lambda}\,(\boldsymbol{t})) \\ m &= 1, 2, \cdots, M \end{aligned} \tag{38}$$

Let us define $\boldsymbol{X}_{\mathrm{u},m,\mathrm{I}}$, $\boldsymbol{G}_{\mathrm{u},m}$, $\boldsymbol{X}_{\mathrm{l},m,\mathrm{I}}$ and $\boldsymbol{G}_{\mathrm{l},m}$ as the link flow and green-time proportion vectors of the m-th OD pair containing all elements in an arbitrary vector $\boldsymbol{Y} = [\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]$ on $\boldsymbol{B}$ which reach their upper and lower boundary, respectively. The rest elements in $\boldsymbol{Y}$ are defined as $\boldsymbol{Y}_{\mathrm{r}}$. Then, from (37), the updated $\boldsymbol{X}_{\mathrm{u},m,\mathrm{I}}$ and $\boldsymbol{G}_{\mathrm{u},m}$ will become smaller if $\boldsymbol{A}_{\mathrm{ux},\boldsymbol{m}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}} \leq \boldsymbol{0}$ and $\boldsymbol{A}_{\mathrm{ug},\boldsymbol{m}}\boldsymbol{\Delta G_m} \leq \boldsymbol{0}$; while the updated $\boldsymbol{X}_{\mathrm{l},m,\mathrm{I}}$ and $\boldsymbol{G}_{\mathrm{l},m}$ will become larger if $\boldsymbol{A}_{\mathrm{lx},\boldsymbol{m}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}} \geq \boldsymbol{0}$ and $\boldsymbol{A}_{\mathrm{lg},\boldsymbol{m}}\boldsymbol{\Delta G_m} \geq \boldsymbol{0}$. Because $t_{n_m,m,\mathrm{I}}\,(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})$ and $P\,(b\,(\boldsymbol{X},\boldsymbol{G}))$ are continuous functions, there exists a positive number λ that lets the updated $\boldsymbol{Y}$

$$\boldsymbol{Y} = \begin{bmatrix} \boldsymbol{X}_{\mathrm{u},m,\mathrm{I}} \\ \boldsymbol{G}_{\mathrm{u},m} \\ \boldsymbol{X}_{\mathrm{l},m,\mathrm{I}} \\ \boldsymbol{G}_{\mathrm{l},m} \\ \boldsymbol{Y}_{\mathrm{r}} \end{bmatrix} + \lambda \begin{bmatrix} \boldsymbol{A}_{\mathrm{ux},\boldsymbol{m}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}} \\ \boldsymbol{A}_{\mathrm{ug},\boldsymbol{m}}\boldsymbol{\Delta G_m} \\ \boldsymbol{A}_{\mathrm{lx},\boldsymbol{m}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}} \\ \boldsymbol{A}_{\mathrm{lg},\boldsymbol{m}}\boldsymbol{\Delta G_m} \\ \boldsymbol{\Delta Y}_{\mathrm{r}} \end{bmatrix} \tag{39}$$

be located in $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$. For any vectors on $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S} - \boldsymbol{B}$, there exists a positive number λ that lets the updated $\boldsymbol{Y}$ be still located in $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ because of the continuity of $t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$ and $P(b(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))$. That means $\boldsymbol{f} : \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S} \to \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$. The set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ is a compact and convex set. Then, from Brouwer's fixed point theorem, there exists a fixed point $\left[\boldsymbol{X}_m^*, \boldsymbol{G}_m^*\right]$ on $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ that lets

$$\left[\boldsymbol{X}_m^*, \boldsymbol{G}_m^*\right] = \boldsymbol{f}\left(\left[\boldsymbol{X}_m^*, \boldsymbol{G}_m^*\right]\right), \quad m = 1, 2, \cdots, M \tag{40}$$

which means that $\boldsymbol{\Delta}\left[\boldsymbol{X}_m^*, \boldsymbol{G}_m^*\right] = \boldsymbol{0}$, and thus $\boldsymbol{X}_m^*$ is the Wardrop equilibrium. □

Remark 4: Note that lemma 3 gives the sufficient condition for existence of Wardrop equilibrium, rather than necessary condition. If $t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$ and $P(b(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))$ are not monotone increasing functions for any link flows, there are possibilities that the Wardrop equilibrium exists although the condition is not fulfilled. In that case, the dynamical system becomes unstable in certain regions. The stability will be discussed in the following context.

Corollary 1: If $t_{n_m,m,I}(\boldsymbol{X}_I, \boldsymbol{G})$ and $P(b(\boldsymbol{X}_I, \boldsymbol{G}))$ are monotone functions of $\boldsymbol{X}_I$ for any link flow and the condition in lemma 3 is fulfilled, then $\boldsymbol{X}_I^*$ is unique.

Proof: Suppose that there is an equilibrium $\left[\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right]$ in $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, which let the travel cost on each input link and the stage pressure on each incident link be equal to each other, that is, $t_{r,m,\mathrm{I}}\left(\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right) = t_{s,m,\mathrm{I}}\left(\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right)$ for any r-th and s-th input links of the m-th OD pair, and $[P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{s,i} = [P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{r,i}$ for any r-th and s-th links incident to the i-th intersection. For an arbitrary $[\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}] \neq \left[\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right]$, it has $t_{r,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \neq t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$ and $[P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{s,i} \neq [P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{r,i}$ because of monotonicity of $t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$ and $P(b(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))$. Then corollary 1 is proved. □

Corollary 1 gives the condition for uniqueness of the equilibrium for a given $\boldsymbol{G}$, i.e., a given feasible set. In the proposed dynamical system, $\boldsymbol{G}$ and $\boldsymbol{X}$ change over time and thus there exists a set of equilibria rather than a unique equilibrium even though $t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$ and $P(b(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))$ are monotone functions. Recall that equation (31) aims at finding a solution minimizing the sum of product of link flow and link cost. Each equilibrium in the set of equilibria gives a link-flow and link-cost vector, and thus gives different product. Then, we can obtain the following lemma:

Lemma 4: The set of equilibria $\boldsymbol{E}$ is a compact, non-convex set. Any point $\left[\boldsymbol{X}_I^*, \boldsymbol{G}^*\right]$ in $\boldsymbol{E}$ satisfies

$$\frac{\partial^{(K)} t_{k,m,I}\left(\boldsymbol{X}_I^*, \boldsymbol{G}^*\right)}{\partial \boldsymbol{X}_I \partial \boldsymbol{G}} = \frac{\partial^{(K)} t_{\ell,m,I}\left(\boldsymbol{X}_I^*, \boldsymbol{G}^*\right)}{\partial \boldsymbol{X}_I \partial \boldsymbol{G}} \qquad k \neq \ell, \quad m = 1, 2, \cdots, M \tag{41}$$

where K is the number of variables.

Proof: From the definition of Wardrop equilibrium, the vector $\left[-\boldsymbol{t}^\mathrm{T}\left(\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right), \boldsymbol{0}\right]$ is normal at $\left[\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right]$ to the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$. Recall that $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ is a subset of an affine space. Thus, for each $\boldsymbol{G}^*$, it has

$$t_{k,m,\mathrm{I}}\left(\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right) = t_{\ell,m,\mathrm{I}}\left(\boldsymbol{X}_\mathrm{I}^*, \boldsymbol{G}^*\right) \qquad k \neq \ell, \quad m = 1, 2, \cdots, M \tag{42}$$

Because $t_{k,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$ is a continuous monotone function on $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$ for a given $\boldsymbol{G}$ for $k = 1, 2, \cdots, N_m$, and the adjustment of $\boldsymbol{G}$ is a continuous process, the solution space for (42) is continuous and a subspace of $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, and thus it is compact. Suppose that $\left[\boldsymbol{X}_{1,\mathrm{I}}^*, \boldsymbol{G}_1^*\right]$ and $\left[\boldsymbol{X}_{2,\mathrm{I}}^*, \boldsymbol{G}_2^*\right]$ are any two equilibria in the solution space, then, an arbitrary point on the line segment between the two points can be expressed by $\left[\boldsymbol{X}_{\mathrm{s},\mathrm{I}}, \boldsymbol{G}_\mathrm{s}\right] = \theta_1\left[\boldsymbol{X}_{1,\mathrm{I}}^*, \boldsymbol{G}_1^*\right] + (1-\theta_1)\left[\boldsymbol{X}_{2,\mathrm{I}}^*, \boldsymbol{G}_2^*\right]$ for $0 \leq \theta_1 \leq 1$. It is obvious that $t_{k,m,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{s},\mathrm{I}}, \boldsymbol{G}_\mathrm{s}\right) \neq t_{\ell,m,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{s},\mathrm{I}}^*, \boldsymbol{G}_\mathrm{s}^*\right)$ for $k \neq \ell, m = 1, 2, \cdots, M$ because of non-linearity of $t_{k,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})$. Thus, the solution space is a non-convex set.

The two equilibria $\left[\boldsymbol{X}_{1,\mathrm{I}}^*, \boldsymbol{G}_1^*\right]$ and $\left[\boldsymbol{X}_{2,\mathrm{I}}^*, \boldsymbol{G}_2^*\right]$ satisfy

$$\int_{\boldsymbol{X}_{1,\mathrm{I}}^*}^{\boldsymbol{X}_{2,\mathrm{I}}^*} \int_{\boldsymbol{G}_1^*}^{\boldsymbol{G}_2^*} \left(\begin{array}{c} \frac{\partial^{(K)} t_{k,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})}{\partial \boldsymbol{X}_\mathrm{I} \partial \boldsymbol{G}} - \\ \frac{\partial^{(K)} t_{\ell,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})}{\partial \boldsymbol{X}_\mathrm{I} \partial \boldsymbol{G}} \end{array} \right) \mathrm{d}\boldsymbol{X}_\mathrm{I} \mathrm{d}\boldsymbol{G} = 0 \qquad k \neq \ell, \quad m = 1, 2, \cdots, M \tag{43}$$

where $\int_{\boldsymbol{X}_{1,\mathrm{I}}^*}^{\boldsymbol{X}_{2,\mathrm{I}}^*} \int_{\boldsymbol{G}_1^*}^{\boldsymbol{G}_2^*}$ represents multiple integrals from $\left[\boldsymbol{X}_{1,\mathrm{I}}^*, \boldsymbol{G}_1^*\right]$ to $\left[\boldsymbol{X}_{2,\mathrm{I}}^*, \boldsymbol{G}_2^*\right]$. Let us define $\left[\boldsymbol{X}_{2,\mathrm{I}}^*, \boldsymbol{G}_2^*\right] = \left[\boldsymbol{X}_{1,\mathrm{I}}^*, \boldsymbol{G}_1^*\right] + [\boldsymbol{\Delta X}_\mathrm{I}, \boldsymbol{\Delta G}_1]$, when $[\boldsymbol{\Delta X}_\mathrm{I}, \boldsymbol{\Delta G}_1]$ is close to zero, (43) becomes

$$\lim_{[\boldsymbol{\Delta X}_\mathrm{I}, \boldsymbol{\Delta G}_1] \to \boldsymbol{0}} \int_{\boldsymbol{X}_{1,\mathrm{I}}^*}^{\boldsymbol{X}_{1,\mathrm{I}}^* + \boldsymbol{\Delta X}_\mathrm{I}} \int_{\boldsymbol{G}_1^*}^{\boldsymbol{G}_1^* + \boldsymbol{\Delta G}_1} \times \left(\frac{\partial^{(K)} t_{k,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})}{\partial \boldsymbol{X}_\mathrm{I} \partial \boldsymbol{G}} - \frac{\partial^{(K)} t_{\ell,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})}{\partial \boldsymbol{X}_\mathrm{I} \partial \boldsymbol{G}} \right) \mathrm{d}\boldsymbol{X}_\mathrm{I} \mathrm{d}\boldsymbol{G} = 0 \qquad k \neq \ell, m = 1, 2, \cdots, M$$

then lemma 4 is proved. □

From the definition in [13], the dynamic system given by (37) is stable if and only if, for any initial vector $\left[\boldsymbol{X}_{\boldsymbol{0},\mathrm{I}}, \boldsymbol{G}_{\boldsymbol{0}}\right] \in \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$, the solution of (37) with the initial vector $\left[\boldsymbol{X}_{\boldsymbol{0},\mathrm{I}}, \boldsymbol{G}_{\boldsymbol{0}}\right]$ converges as $t \to \infty$, to the set of equilibria $\boldsymbol{E}$. To prove the stability of the dynamical system, let us define a function V by

$$V([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]) = \sum_{m=1}^{M} V_{X,m}([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]) + \sum_{i=1}^{I} V_{G,i}([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]) \tag{44}$$

where

$$\begin{aligned} V_{X,m}([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]) &= \sum_{r,s=1,r\neq s}^{N_m} X_{1,r,m} \left(\begin{array}{c} t_{r,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) - \\ t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{array} \right)_+^2 \\ &= \boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^\mathrm{T}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \boldsymbol{X}_{\boldsymbol{m},\mathrm{I}} \\ V_{G,i}([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]) &= \sum_{r,s=1,r\neq s}^{N_i} G_{r,i} \left(\begin{array}{c} [P(\boldsymbol{b}(\boldsymbol{X}, \boldsymbol{G}))]_{s,i} - \\ [P(\boldsymbol{b}(\boldsymbol{X}, \boldsymbol{G}))]_{r,i} \end{array} \right)_+^2 \\ &= \boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}^\mathrm{T}(\boldsymbol{X}, \boldsymbol{G}) \boldsymbol{G}_{\boldsymbol{i}} \end{aligned}$$

where N_m and N_i are respectively the number of routes for the m-th OD pair and the number of links incident to the i-th intersection, $\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}} = \left[X_{1,1,m} \; X_{1,2,m} \; \cdots \; X_{1,N_m,m} \right]^\mathrm{T}$, $\boldsymbol{G}_{\boldsymbol{i}} =$

$$\left[\, G_{1,i}\ \ G_{2,i}\ \cdots\ G_{N_i,i}\,\right]^{\mathrm{T}},$$

$$\boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)=\left[\sum_{s=2}^{N_m}\begin{pmatrix} t_{1,m,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)- \\ t_{s,m,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\end{pmatrix}_{+}^{2}\cdots \sum_{s=1,s\neq N_m}^{N_m}\begin{pmatrix} t_{N_m,m,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)- \\ t_{s,m,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\end{pmatrix}_{+}^{2}\right]^{\mathrm{T}} \tag{45}$$

$$\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}\left(\boldsymbol{X},\boldsymbol{G}\right)=\left[\sum_{s=2}^{N_i}\begin{pmatrix} \left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{s,i}- \\ \left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{1,i}\end{pmatrix}_{+}^{2}\cdots \sum_{s=1,s\neq N_i}^{N_i}\begin{pmatrix} \left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{s,i}- \\ \left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{N_i,i}\end{pmatrix}_{+}^{2}\right]^{\mathrm{T}} \tag{46}$$

It is obvious that $V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)$=0 if and only if $[\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]$ is an equilibrium and $V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)>0$ for all $[\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\in\boldsymbol{D}\times\boldsymbol{F}\cap\boldsymbol{S}\backslash\boldsymbol{E}$, where $\boldsymbol{A}\backslash\boldsymbol{B}$ represents the relative complement of $\boldsymbol{A}$ with respect to $\boldsymbol{B}$. Now we investigate monotonicity of $V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)$ with t. The time derivative of $V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)$ can be expressed by

$$\begin{aligned}
&\frac{dV\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{dt}\\
&=\frac{\partial V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial[\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]}\frac{d[\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]}{dt}\\
&=\frac{\partial V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{X}_{\mathrm{I}}}\frac{d\boldsymbol{X}_{\mathrm{I}}}{dt}+\frac{\partial V\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{G}}\frac{d\boldsymbol{G}}{dt}\\
&=\sum_{m=1}^{M}\left(\sum_{n=1}^{M}\frac{\partial V_{X,n}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}+\sum_{i=1}^{I}\frac{\partial V_{G,i}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}\right)\frac{d\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}{dt}\\
&\quad+\sum_{i=1}^{I}\left(\sum_{j=1}^{I}\frac{\partial V_{G,j}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{G}_{\boldsymbol{i}}}+\sum_{n=1}^{M}\frac{\partial V_{X,n}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{G}_{\boldsymbol{i}}}\right)\frac{d\boldsymbol{G}_{\boldsymbol{i}}}{dt}\\
&=\sum_{m=1}^{M}\Bigg(\boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)+\sum_{n=1}^{M}\boldsymbol{X}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\frac{\partial\boldsymbol{C}_{\mathrm{d},\boldsymbol{n},\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}\\
&\quad+\sum_{i=1}^{I}\left(\boldsymbol{G}_{\boldsymbol{i}}^{\mathrm{T}}\frac{\partial\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}\right)\Bigg)\frac{d\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}{dt}\\
&\quad+\sum_{i=1}^{I}\Bigg(\sum_{n=1}^{M}\left(\boldsymbol{X}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\frac{\partial\boldsymbol{C}_{\mathrm{d},\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{G}_{\boldsymbol{i}}}\right)\\
&\quad+\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)+\sum_{j=1}^{I}\boldsymbol{G}_{\boldsymbol{j}}^{\mathrm{T}}\frac{\partial\boldsymbol{P}_{\mathrm{d},\boldsymbol{j}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{G}_{\boldsymbol{i}}}\Bigg)\frac{d\boldsymbol{G}_{\boldsymbol{i}}}{dt}
\end{aligned} \tag{47}$$

By (45) and (46), $\boldsymbol{X}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\frac{\partial\boldsymbol{C}_{\mathrm{d},\boldsymbol{n},\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}$, $\boldsymbol{G}_{\boldsymbol{i}}^{\mathrm{T}}\frac{\partial\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}$, $\boldsymbol{X}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\frac{\partial\boldsymbol{C}_{\mathrm{d},\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial\boldsymbol{G}_{\boldsymbol{i}}}$ and $\boldsymbol{G}_{\boldsymbol{j}}^{\mathrm{T}}\frac{\partial\boldsymbol{P}_{\mathrm{d},\boldsymbol{j}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial\boldsymbol{G}_{\boldsymbol{i}}}$ can be determined by

$$\begin{aligned}
\boldsymbol{X}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\frac{\partial\boldsymbol{C}_{\mathrm{d},\boldsymbol{n},\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}&=\left[\,CX_{\mathrm{d},1}\ \ CX_{\mathrm{d},2}\ \cdots\ CX_{\mathrm{d},N_m}\,\right]\\
&=-\frac{2}{w}\boldsymbol{\Delta X}_{\boldsymbol{n},\mathrm{I}}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\\
CX_{\mathrm{d},n_m}&=2\sum_{\ell=1}^{N_n}X_{1,\ell,n}\sum_{s=1,s\neq\ell}^{N_n}\begin{pmatrix} t_{\ell,n,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)- \\ t_{s,n,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\end{pmatrix}_{+}\\
&\quad\times\begin{pmatrix}\frac{\partial t_{\ell,n,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial X_{1,n_m,m}}- \\ \frac{\partial t_{s,n,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial X_{1,n_m,m}}\end{pmatrix}\\
\boldsymbol{G}_{\boldsymbol{i}}^{\mathrm{T}}\frac{\partial\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}&=\left[\,PX_{\mathrm{d},1}\ \ PX_{\mathrm{d},2}\ \cdots\ PX_{\mathrm{d},N_m}\,\right]\\
&=\frac{2}{w}\boldsymbol{\Delta G}_{i,\mathrm{GP}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{px},\boldsymbol{im}}\\
PX_{\mathrm{d},n_m}&=2\sum_{\ell=1}^{N_i}G_{\ell,i}\sum_{s=1,s\neq\ell}^{N_i}\\
&\quad\times\begin{pmatrix}\left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{s,i}- \\ \left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{\ell,i}\end{pmatrix}_{+}\\
&\quad\times\begin{pmatrix}P'\left(b_{s,i}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\frac{\partial b_{s,i}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial X_{1,n_m,m}}- \\ P'\left(b_{\ell,i}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\frac{\partial b_{\ell,i}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial X_{1,n_m,m}}\end{pmatrix}\\
\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}\frac{\partial\boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{G}_{\boldsymbol{i}}}&=\left[\,CG_{\mathrm{d},1}\quad CG_{\mathrm{d},2}\quad\cdots\quad CG_{\mathrm{d},N_i}\,\right]\\
&=-\frac{2}{w}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cg},\boldsymbol{mi}}\\
CG_{\mathrm{d},n_i}&=2\sum_{\ell=1}^{N_m}X_{1,\ell,m}\sum_{s=1,s\neq\ell}^{N_m}\begin{pmatrix}t_{\ell,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})- \\ t_{s,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})\end{pmatrix}_{+}\\
&\quad\times\begin{pmatrix}\frac{\partial t_{\ell,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial G_{n_i,i}}- \\ \frac{\partial t_{s,m,\mathrm{I}}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial G_{n_i,i}}\end{pmatrix}\\
\boldsymbol{G}_{\boldsymbol{j}}^{\mathrm{T}}\frac{\partial\boldsymbol{P}_{\mathrm{d},\boldsymbol{j}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial\boldsymbol{G}_{\boldsymbol{i}}}&=\left[\,PG_{\mathrm{d},1}\quad PG_{\mathrm{d},2}\quad\cdots\quad PG_{\mathrm{d},N_i}\,\right]\\
&=\frac{2}{w}\boldsymbol{\Delta G}_{i,\mathrm{GP}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{pg},\boldsymbol{ji}}\\
PG_{\mathrm{d},n_i}&=2\sum_{\ell=1}^{N_j}G_{\ell,j}\sum_{s=1,s\neq\ell}^{N_j}\\
&\quad\times\begin{pmatrix}\left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{s,j}- \\ \left[P\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\right]_{\ell,j}\end{pmatrix}_{+}\\
&\quad\times\begin{pmatrix}P'\left(b_{s,j}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)\frac{\partial b_{s,j}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial G_{n_i,i}}- \\ P'\left(b_{\ell,j}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})\right)\frac{\partial b_{\ell,j}(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G})}{\partial G_{n_i,i}}\end{pmatrix}
\end{aligned} \tag{48}$$

where $\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}$, $\boldsymbol{J}_{\mathrm{px},\boldsymbol{im}}$, $\boldsymbol{J}_{\mathrm{cg},\boldsymbol{mi}}$ and $\boldsymbol{J}_{\mathrm{pg},\boldsymbol{ji}}$ are the Jacobian matrices of $\boldsymbol{t}_{\mathrm{I},n}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)$, $\boldsymbol{P}_{\boldsymbol{i}}\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)$, $\boldsymbol{t}_{\mathrm{I},\boldsymbol{m}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)$ and $\boldsymbol{P}_{\boldsymbol{j}}\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)$ evaluated at $\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}$, $\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}$, $\boldsymbol{G}_{\boldsymbol{i}}$ and $\boldsymbol{G}_{\boldsymbol{i}}$, respectively. Thus,

$$\begin{aligned}
&\left(\sum_{n=1}^{M}\frac{\partial V_{X,n}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}+\sum_{i=1}^{I}\frac{\partial V_{G,i}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}\right)\frac{d\boldsymbol{X}_{\boldsymbol{m},\mathrm{I}}}{dt}\\
&=-\frac{2}{w}\sum_{n=1}^{M}\boldsymbol{\Delta X}_{\boldsymbol{n},\mathrm{I}}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}+\boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}\\
&\quad+\frac{2}{w}\sum_{i=1}^{I}\boldsymbol{\Delta G}_{i,\mathrm{GP}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{px},\boldsymbol{im}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}=\boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}\\
&\quad-\frac{2}{w}\boldsymbol{\Delta X}^{\mathbf{T}}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}\boldsymbol{\Delta X}
\end{aligned}$$

$$+\frac{2}{w}\boldsymbol{\Delta G}^{\mathrm{T}}\sum_{i=1}^{I}\boldsymbol{B}_{\boldsymbol{i}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{px},\boldsymbol{im}}\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}\boldsymbol{\Delta X} \tag{49}$$

and

$$\begin{aligned}&\left(\sum_{j=1}^{I}\frac{\partial V_{G,j}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial \boldsymbol{G_i}}+\sum_{n=1}^{M}\frac{\partial V_{X,n}\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{\partial \boldsymbol{G_i}}\right)\frac{d\boldsymbol{G_i}}{dt}\\&\quad=-\frac{2}{w}\sum_{n=1}^{M}\boldsymbol{\Delta X}_{\boldsymbol{n},\mathrm{I}}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cg},\boldsymbol{ni}}\boldsymbol{\Delta G}_{i,\mathrm{GP}}+\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}^{\mathrm{T}}\left(\boldsymbol{X},\boldsymbol{G}\right)\boldsymbol{\Delta G}_{i,\mathrm{GP}}\\&\qquad+\frac{2}{w}\sum_{j=1}^{I}\boldsymbol{\Delta G}_{j,\mathrm{GP}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{pg},\boldsymbol{ji}}\boldsymbol{\Delta G}_{i,\mathrm{GP}}=\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}^{\mathrm{T}}\left(\boldsymbol{X},\boldsymbol{G}\right)\boldsymbol{\Delta G}_{i,\mathrm{GP}}\\&\qquad-\frac{2}{w}\boldsymbol{\Delta X}^{\mathbf{T}}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{cg},\boldsymbol{ni}}\boldsymbol{B_i}\boldsymbol{\Delta G}\\&\qquad+\frac{2}{w}\boldsymbol{\Delta G}^{\mathrm{T}}\sum_{j=1}^{I}\boldsymbol{B}_{\boldsymbol{j}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{pg},\boldsymbol{ji}}\boldsymbol{B_i}\boldsymbol{\Delta G}\end{aligned} \tag{50}$$

where $\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}$ and $\boldsymbol{B_i}$ are the scaling matrices for input links of the m-th OD pair and the i-th intersection, respectively. Combined with (49) and (50), (47) can be rewritten by

$$\begin{aligned}&\frac{dV\left([\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}]\right)}{dt}\\&\quad=\sum_{m=1}^{M}\boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}\\&\qquad+\sum_{i=1}^{I}\boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}^{\mathrm{T}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\boldsymbol{\Delta G}_{i,\mathrm{GP}}\\&\qquad-\frac{2}{w}\boldsymbol{\Delta X}^{\mathbf{T}}\sum_{m=1}^{M}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}\boldsymbol{\Delta X}\\&\qquad+\frac{2}{w}\boldsymbol{\Delta G}^{\mathrm{T}}\sum_{i=1}^{I}\sum_{j=1}^{I}\boldsymbol{B}_{\boldsymbol{j}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{pg},\boldsymbol{ji}}\boldsymbol{B_i}\boldsymbol{\Delta G}\\&\qquad+\frac{2}{w}\boldsymbol{\Delta X}^{\mathbf{T}}\sum_{n=1}^{M}\sum_{i=1}^{I}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{J}_{\mathrm{px},\boldsymbol{in}}^{\mathbf{T}}-\boldsymbol{J}_{\mathrm{cg},\boldsymbol{ni}}\right)\boldsymbol{B_i}\boldsymbol{\Delta G}\end{aligned} \tag{51}$$

The stage pressure function $\boldsymbol{P_i}\left(\boldsymbol{b}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)\right)$ is a non-increasing monotone function of $\boldsymbol{G}$, $\boldsymbol{J}_{\mathrm{pg}}=\sum_{i=1}^{I}\sum_{j=1}^{I}\boldsymbol{B}_{\boldsymbol{j}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{pg},\boldsymbol{ji}}\boldsymbol{B_i}$ is a diagonal matrix, and thus it is a negative semi-definite matrix. The travel cost function $\boldsymbol{t}_{n,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)$ is non-decreasing monotone function of $\boldsymbol{X}_{\mathrm{I}}$. However, $\sum_{m=1}^{M}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}$ is non-symmetric and thus it is cumbersome to state that $\sum_{m=1}^{M}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}$ is a positive semi-definite matrix. We propose the following theorem to determine the matrix definiteness.

Theorem 1: A square matrix $\boldsymbol{M}$ is positive (or negative) semi-definite over a given compact space $\boldsymbol{S}$, i.e., $\boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}\geq 0$ $\left(or\ \boldsymbol{x}^T\boldsymbol{M}\boldsymbol{x}\leq 0\right)$ for any $\boldsymbol{x}\in\boldsymbol{S}$ iff (if and only if) $\boldsymbol{M}$ is a full-rank matrix and for any element $\boldsymbol{x_B}\in\boldsymbol{B_S}$, where $\boldsymbol{B_S}$ is the boundary of $\boldsymbol{S}$ and satisfies $\boldsymbol{x}_{\boldsymbol{B}}^T\boldsymbol{M}\boldsymbol{x_B}\geq 0$ $\left(or\ \boldsymbol{x}_{\boldsymbol{B}}^T\boldsymbol{M}\boldsymbol{x_B}\leq 0\right)$.

Proof: Let us define $y=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x}$ for $\boldsymbol{x}\in\boldsymbol{S}$. By the definition of critical point that a critical point a differential function is any value in its domain where its derivative is zero [26]. Then, the critical point $\boldsymbol{x}_{\mathrm{m}}$ can be determined by

$$\frac{\partial y}{\partial \boldsymbol{x}}=\boldsymbol{x}^{\mathrm{T}}\boldsymbol{M}=\mathbf{0}$$

When $\boldsymbol{M}$ is a full-rank matrix, $\boldsymbol{x}_{\mathrm{m}}=\mathbf{0}$ becomes the unique critical point. It is straightforward that $y=0$ at the critical point. Suppose that for any element $\boldsymbol{x_B}\in\boldsymbol{B_S}$, where $\boldsymbol{B_S}$ is the boundary of $\boldsymbol{S}$ and satisfies $\boldsymbol{x}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x_B}\geq 0$ (or $\boldsymbol{x}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x_B}\leq 0$). We hypothesize that there exists at least one point within $\boldsymbol{S}$ leading to $y<0$ (or $y>0$), then there are at least one critical point $\boldsymbol{x}_{\mathrm{c}}\in\boldsymbol{S}$, that lets $y<0$ (or $y>0$), which is conflict with the uniqueness of $\boldsymbol{x}_{\mathrm{m}}$. Then, theorem 1 is proved. □

From (33), (34) and (36), $\boldsymbol{\Delta X}$ fulfills $\mathbf{1}^{\mathbf{T}}\boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}}=0$, $-T_m\leq \Delta X_{l,m,\mathrm{I}}\leq T_m$. Let us define $\boldsymbol{S_{\Delta X}}$ and $\boldsymbol{S_{\Delta G}}$ as the compact space for $\boldsymbol{\Delta X}$ and $\boldsymbol{\Delta G}$, respectively, $\boldsymbol{B_{\Delta X}}$ and $\boldsymbol{B_{\Delta G}}$ as their boundaries, respectively. Then, with theorem 1, we can get the following corollary:

Corollary 2: The matrix $\sum_{m=1}^{M}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},I}^{T}\boldsymbol{J}_{cx,\boldsymbol{nm}}\boldsymbol{A}_{\boldsymbol{m},I}$ is positive semi-definite over $\boldsymbol{S_{\Delta X}}$.

Proof: For simplicity of expression, let us express $\sum_{m=1}^{M}\sum_{n=1}^{M}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\boldsymbol{J}_{\mathrm{cx},\boldsymbol{nm}}\boldsymbol{A}_{\boldsymbol{m},\mathrm{I}}$ by $\boldsymbol{J}_{\mathrm{cx}}$. Then $\boldsymbol{\Delta X}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cx}}\boldsymbol{\Delta X}=\boldsymbol{\Delta X}_{\mathbf{I}}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cx,I}}\boldsymbol{\Delta X}_{\mathbf{I}}$, where $\boldsymbol{J}_{\mathrm{cx,I}}$ is the $\sum_{m=1}^{M}N_m\times\sum_{m=1}^{M}N_m$ Jacobian matrix. From (33), (34), we can know that $\boldsymbol{J}_{\mathrm{cx,I}}$ is a full-rank matrix, and the element on the k-th row and ℓ-th column can be expressed by

$$\begin{aligned}J_{k\ell,\mathrm{cx,I}}&=\frac{\partial t_{k,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)}{\partial X_{\ell,\mathrm{I}}}=\frac{\partial\left(\boldsymbol{P}_{\boldsymbol{k}}'^{\mathrm{T}}\boldsymbol{C}+\boldsymbol{P}_{\boldsymbol{k}}'^{\mathrm{T}}\boldsymbol{b}\left(\boldsymbol{P}'X_{\mathrm{I}},\boldsymbol{G}\right)\right)}{\partial X_{\ell,\mathrm{I}}}\\&=\boldsymbol{P}_{\boldsymbol{k}}'^{\mathrm{T}}\frac{\partial\boldsymbol{b}\left(\boldsymbol{P}'X_{\mathrm{I}},\boldsymbol{G}\right)}{\partial X_{\ell,\mathrm{I}}}\end{aligned} \tag{52}$$

The space $\boldsymbol{S_{\Delta X}}$ is a subset of affine space. The vertices on $\boldsymbol{B_{\Delta X}}$ are $\sum_{m=1}^{M}N_m\times 1$ vectors $\left[\mathbf{0}\ -T_m\ \mathbf{0}\ T_m\ \mathbf{0}\right]^{\mathbf{T}}$, where the length of first $\mathbf{0}$, second $\mathbf{0}$ and third $\mathbf{0}$ are respectively $\sum_{m'=1}^{m-1}N_{m'}+k_{1,m}$, $k_{2,m}$ and $\sum_{m'=m+1}^{M}N_{m'}+k_{3,m}$, and $N_m=2+k_{1,m}+k_{2,m}+k_{3,m}$. There are $\sum_{m=1}^{M}\binom{2}{N_m}$ vertices on $\boldsymbol{B_{\Delta X}}$. Furthermore, for each vertex $\boldsymbol{\Delta X_v}$, it can be shown that $\boldsymbol{\Delta X}_{\boldsymbol{v}}^{\mathbf{T}}\boldsymbol{J}_{\mathrm{cx,I}}\boldsymbol{\Delta X_v}\geq 0$ from (52) and the fact that $\boldsymbol{t}_{n,\mathrm{I}}\left(\boldsymbol{X}_{\mathrm{I}},\boldsymbol{G}\right)$ is a non-decreasing monotone function of $\boldsymbol{X}_{\mathrm{I}}$. Then for each $\boldsymbol{x_B}\in\boldsymbol{B_{\Delta X}}$, it has $\boldsymbol{x}_{\boldsymbol{B}}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{x_B}\geq 0$, and thus from theorem 1, corollary 2 is proved. □

Corollary 2 shows that the third term in (51) is negative, which leads to the following corollary that shows the summation of last three terms in (51) to be also negative. Let us express $\sum_{n=1}^{M}\sum_{i=1}^{I}\boldsymbol{A}_{\boldsymbol{n},\mathrm{I}}^{\mathrm{T}}\left(\boldsymbol{J}_{\mathrm{px},\boldsymbol{in}}^{\mathbf{T}}-\boldsymbol{J}_{\mathrm{cg},\boldsymbol{ni}}\right)\boldsymbol{B_i}$ by $\boldsymbol{J}_{\mathrm{xg}}$, then the following corollary can be obtained:

Corollary 3: The derivative $dV\left([\boldsymbol{X_I},\boldsymbol{G}]\right)/dt$ fulfills the following inequality

$$\begin{aligned}\frac{dV\left([\boldsymbol{X_I},\boldsymbol{G}]\right)}{dt}\leq&\sum_{m=1}^{M}\boldsymbol{C}_{d,\boldsymbol{m},I}^{T}\left(\boldsymbol{X_I},\boldsymbol{G}\right)\boldsymbol{\Delta X}_{\boldsymbol{m},I}\\&+\sum_{i=1}^{I}\boldsymbol{P}_{d,\boldsymbol{i}}^{T}\left(\boldsymbol{X_I},\boldsymbol{G}\right)\boldsymbol{\Delta G}_{i,GP}\end{aligned} \tag{53}$$

Proof: Equation (51) can be rewritten by

$$\begin{aligned}\frac{dV([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}])}{dt} &= \sum_{m=1}^{M} \boldsymbol{C}_{\mathrm{d},\boldsymbol{m},\mathrm{I}}^{\mathrm{T}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G})\, \boldsymbol{\Delta X}_{\boldsymbol{m},\mathrm{I}} \\ &\quad + \sum_{i=1}^{I} \boldsymbol{P}_{\mathrm{d},\boldsymbol{i}}^{\mathrm{T}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \boldsymbol{\Delta G}_{i,\mathrm{GP}} \\ &\quad - \frac{2}{w}\left(\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{cx}}' \boldsymbol{\Delta X} + \boldsymbol{\Delta G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G} - \boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{\Delta G}\right)\end{aligned} \tag{54}$$

where $\boldsymbol{J}_{\mathrm{pg}}' = -\boldsymbol{J}_{\mathrm{pg}}$ and $\boldsymbol{J}_{\mathrm{cx}}' = \frac{1}{2}\left(\boldsymbol{J}_{\mathrm{cx}} + \boldsymbol{J}_{\mathrm{cx}}^{\mathbf{T}}\right)$. Recall that $\boldsymbol{J}_{\mathrm{pg}}$ is a negative semi-definite matrix and $\boldsymbol{J}_{\mathrm{cx}}$ is a non-symmetric positive semi-definite matrix over $\boldsymbol{S_{\Delta X}}$. Thus, $\boldsymbol{J}_{\mathrm{pg}}'$ and $\boldsymbol{J}_{\mathrm{cx}}'$ are positive semi-definite matrices. From Cauchy-Schwarz inequality, we have

$$|\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{\Delta G}| \leq \sqrt{\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{H}_{\mathbf{1}}^{\mathrm{T}} \boldsymbol{H}_{\mathbf{1}} \boldsymbol{\Delta X}} \sqrt{\boldsymbol{\Delta G}^{\mathbf{T}} \boldsymbol{H}_{\mathbf{2}}^{\mathrm{T}} \boldsymbol{H}_{\mathbf{2}} \boldsymbol{\Delta G}} \tag{55}$$

where $\boldsymbol{J}_{\mathrm{xg}} = \boldsymbol{H}_{\mathbf{1}}^{\mathrm{T}} \boldsymbol{H}_{\mathbf{2}}$. Then

$$\begin{aligned}&\left(\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{cx}}' \boldsymbol{\Delta X} + \boldsymbol{\Delta G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G}\right)^2 \\ &\quad = \left(\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{cx}}' \boldsymbol{\Delta X}\right)^2 + \left(\boldsymbol{\Delta G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G}\right)^2 \\ &\quad\quad + 2\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{cx}}' \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G}\end{aligned} \tag{56}$$

We let $\boldsymbol{H}_{\mathbf{2}} = \boldsymbol{J}_{\mathrm{pg}}'^{\frac{1}{2}}$, which is a diagonal matrix. Furthermore, $\boldsymbol{H}_{\mathbf{1}}^{\mathbf{T}}$ in (55) becomes $\boldsymbol{H}_{\mathbf{1}}^{\mathrm{T}} = \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{J}_{\mathrm{pg}}'^{-\frac{1}{2}}$. Then, (55) can be transformed to

$$\left(\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{\Delta G}\right)^2 \leq \boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{J}_{\mathrm{pg}}'^{-1} \boldsymbol{J}_{\mathrm{xg}}^{\mathrm{T}} \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G} \tag{57}$$

and we have

$$\begin{aligned}&2\boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{cx}}' \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G} - \\ &\quad \boldsymbol{\Delta X}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{J}_{\mathrm{pg}}'^{-1} \boldsymbol{J}_{\mathrm{xg}}^{\mathrm{T}} \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{pg}}' \boldsymbol{\Delta G} = \boldsymbol{Tr}\left(\boldsymbol{\Delta G} \boldsymbol{\Delta X}^{\mathbf{T}} \times \right. \\ &\quad \left.\left(2\boldsymbol{J}_{\mathrm{cx}}' \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathrm{T}} \boldsymbol{J}_{\mathrm{pg}}' - \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{J}_{\mathrm{pg}}'^{-1} \boldsymbol{J}_{\mathrm{xg}}^{\mathrm{T}} \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{pg}}'\right)\right) = \\ &\quad \boldsymbol{Tr}\left(\boldsymbol{\Delta G} \boldsymbol{\Delta X}^{\mathbf{T}} \left(2\boldsymbol{J}_{\mathrm{cx}}' - \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{J}_{\mathrm{pg}}'^{-1} \boldsymbol{J}_{\mathrm{xg}}^{\mathrm{T}}\right) \boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathbf{T}} \boldsymbol{J}_{\mathrm{pg}}'\right)\end{aligned} \tag{58}$$

The matrix $\boldsymbol{\Delta X} \boldsymbol{\Delta G}^{\mathbf{T}}$ in (58) is with the rank of one, and from the definition of travel cost function and stage pressure vector, $\boldsymbol{J}_{\mathrm{pg}}' \left(2\boldsymbol{J}_{\mathrm{cx}}' - \boldsymbol{J}_{\mathrm{xg}} \boldsymbol{J}_{\mathrm{pg}}'^{-1} \boldsymbol{J}_{\mathrm{xg}}^{\mathrm{T}}\right)$ can be shown to be a positive semi-definite matrix, and thus (58) is a non-negative number. From (55) to (58), the last term in (54) is a non-positive number and thus corollary 3 is proved. □

Then, we can get the following lemma:

Lemma 5: The derivative $\frac{dV([\boldsymbol{X}_I, \boldsymbol{G}])}{dt} < 0$ *for all* $[\boldsymbol{X}_I, \boldsymbol{G}] \in \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S} \backslash \boldsymbol{E}$

Proof: From the definition of the input flow swap vector (32) and the green-time proportion swap vector (36), (53) can be transformed to

$$\begin{aligned}&\frac{dV([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}])}{dt} \\ &\leq w \sum_{m=1}^{M} \sum_{n_m=1}^{N_m} \sum_{s=1, s\neq n_m}^{N_m} \begin{pmatrix} t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \\ -t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+^2 \\ &\quad \times \left(\sum_{s=1, s\neq n_m}^{N_m} (-X_{1,n_m,m}) \begin{pmatrix} t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) - \\ t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+ \right. \\ &\quad \left. + X_{1,s,m} \begin{pmatrix} t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) - \\ t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+ \right) + w \sum_{i=1}^{I} \sum_{n_i=1}^{N_i} \sum_{s=1, s\neq n_i}^{N_i} \\ &\quad \times \begin{pmatrix} [P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{s,i} - \\ [P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{n_i,i} \end{pmatrix}_+^2 \left(\sum_{s=1, s\neq n_i}^{N_i} (-G_{n_i,i}) \right. \\ &\quad \times \begin{pmatrix} [P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{s,i} \\ -[P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{n_i,i} \end{pmatrix}_+ \\ &\quad \left. + G_{s,i} \begin{pmatrix} [P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{n_i,i} \\ -[P(\boldsymbol{b}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}))]_{s,i} \end{pmatrix}_+ \right)\end{aligned} \tag{59}$$

Because of

$$\begin{aligned}0 &\leq \sum_{n_m=1}^{N_m} \sum_{s=1, s\neq n_m}^{N_m} \begin{pmatrix} t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \\ -t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+^2 \\ &\quad \times \sum_{s=1, s\neq n_m}^{N_m} X_{1,s,m} \begin{pmatrix} t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) - \\ t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+ \\ &\leq \sum_{n_m=1}^{N_m} \sum_{s=1, s\neq n_m}^{N_m} \begin{pmatrix} t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \\ -t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+^2 \\ &\quad \times \sum_{s=1, s\neq n_m}^{N_m} X_{1,n_m,m} \begin{pmatrix} t_{n_m,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) - \\ t_{s,m,\mathrm{I}}(\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}) \end{pmatrix}_+\end{aligned}$$

and similarity for the stage pressure function, Lemma 5 is proved. □

Now, we has proved that $V([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}])$=0 if and only if $[\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]$ is an equilibrium, $V([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}]) > 0$ for all $[\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}] \in \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S} \backslash \boldsymbol{E}$ and $\frac{dV([\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}])}{dt} < 0$ for all $[\boldsymbol{X}_\mathrm{I}, \boldsymbol{G}] \in \boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S} \backslash \boldsymbol{E}$. By Lyapunov theorem, we can conclude that the dynamical system given by (37) can converge to an equilibrium on $\boldsymbol{E}$ as time goes to infinity by using the novel control policy, or the P0 control policy if the two conditions proposed in the subsection III-C are satisfied.

V. Numerical Results

The proposed dynamical system given by (37) is tested on an one-OD two-route network illustrated by Fig. 1 and a two-OD two-route network illustrated by Fig. 2.

In order to validate the superiority of our proposed novel control policy over the P0 control policy, we start from an arbitrary feasible point fulfilling the condition given by (1), (2) and (3), set the stage pressure vector to $\boldsymbol{P}(\boldsymbol{b}) = \boldsymbol{k} \bullet (\boldsymbol{C} + \boldsymbol{b})$ and $\boldsymbol{P}(\boldsymbol{b}) = \boldsymbol{s} \bullet \boldsymbol{b}$, respectively, and observe if the dynamical system can converge to a feasible point as time goes to infinity. In order to validate the condition of existence of equilibrium point proposed by Lemma 3, we start from an arbitrary feasible

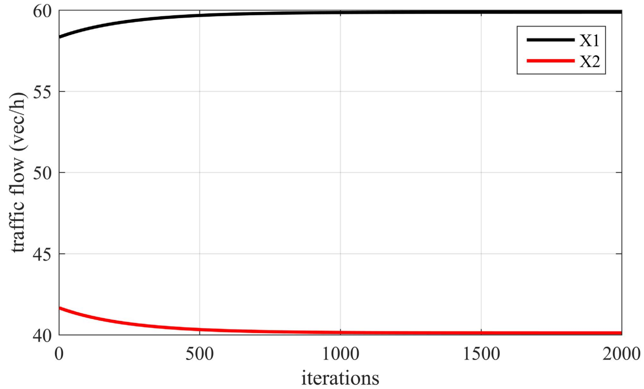

Fig. 4. Time evolution of traffic flow using novel control policy.

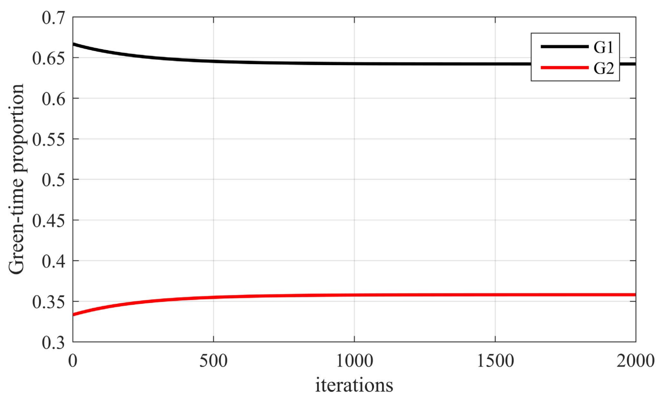

Fig. 5. Time evolution of green-time proportion using novel control policy.

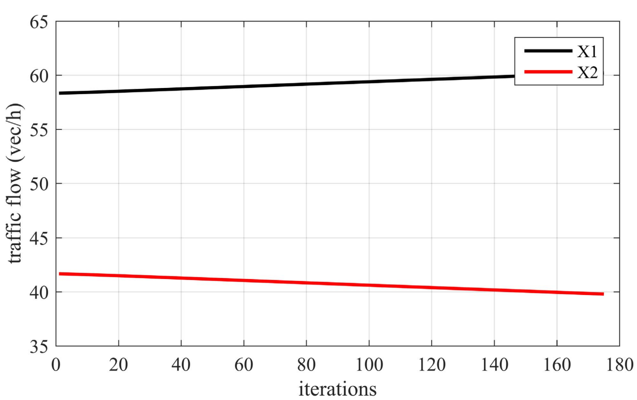

Fig. 6. Time evolution of traffic flow using P0 control policy.

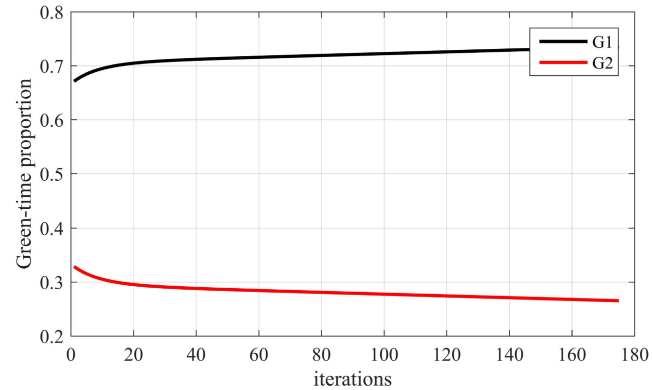

Fig. 7. Time evolution of green-time proportion using P0 control policy.

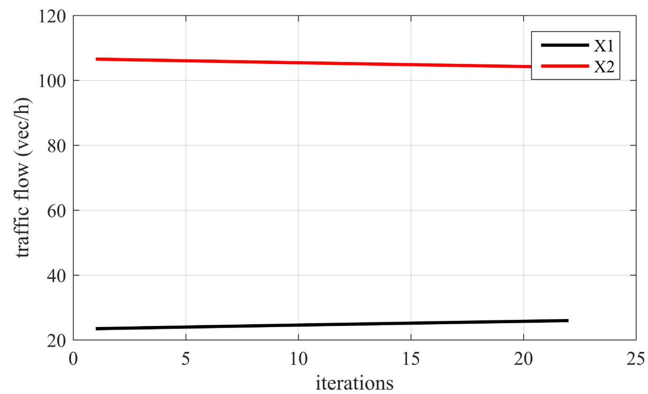

Fig. 8. Time evolution of traffic flow using novel control policy; the condition of existence of equilibrium is NOT fulfilled.

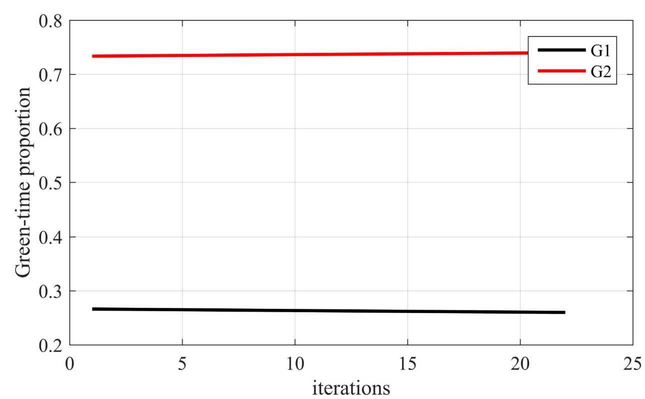

Fig. 9. Time evolution of green-time proportion using novel control policy; the condition of existence of equilibrium is NOT fulfilled.

point for the cases that the boundary condition in Lemma 3 is satisfied or not, and compare the output of the dynamical system.

For the one-OD two-route network, we set the length of two routes and saturation flow of two routes as 0.5km, 1km and 100 vehicles/h, 150 vehicles/h, respectively. The free speed of both routes, the total demand and the duration of traffic light cycle are set to be 50km/h, 100 vehicles/h and 120 seconds, respectively. We start from an arbitrary feasible point $\boldsymbol{XV_0} = [\mathbf{58.33}\ \mathbf{41.67}]$, $\boldsymbol{GV_0} = \left[\mathbf{0.67}\ \mathbf{0.33}\right]$ and set the step length by the rules proposed in [25]. The output of the dynamical system is illustrated by Fig. 4 and Fig. 5, which show the time evolution of traffic flow and green-time proportion on two routes, respectively, when the novel control policy is used. The dynamical system is stable as the equilibrium point can be approached after approximately 1000 iterations. Fig. 6 and Fig. 7 illustrate the output of the dynamical system by using the P0 control policy. It can be seen that traffic flow and green-time proportion cannot approach a feasible equilibrium point

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

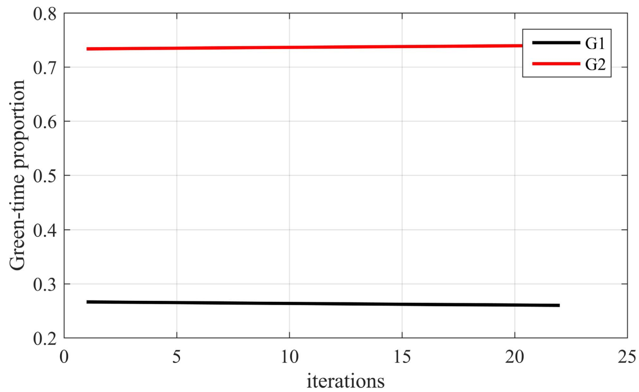

Fig. 10. Time evolution of green-time proportion using novel control policy; the condition of existence of equilibrium is NOT fulfilled.

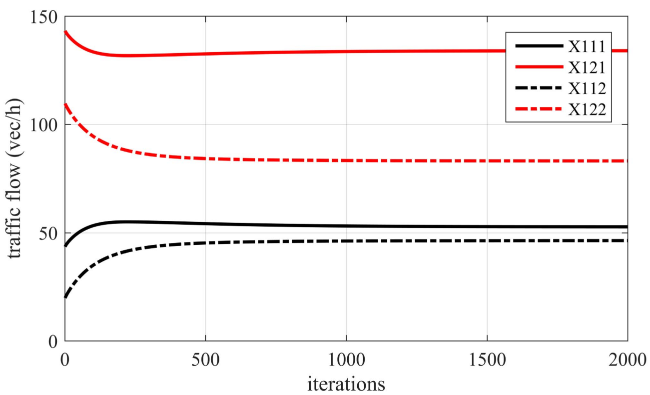

Fig. 11. Time evolution of traffic flow using novel control policy on two-OD two-route network; the condition of existence of equilibrium is fulfilled.

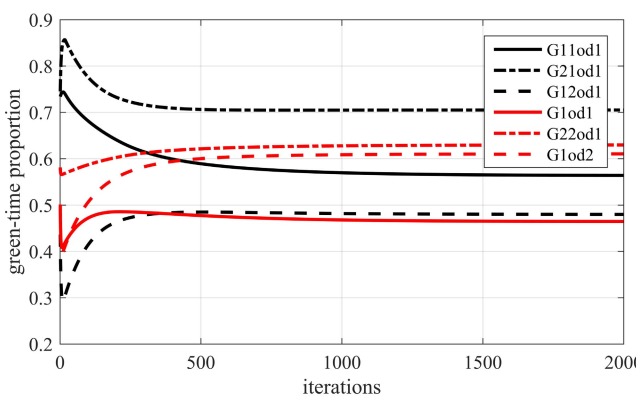

Fig. 12. Time evolution of green-time proportion using novel control policy on two-OD two-route network; the condition of existence of equilibrium is fulfilled.

within the first 180 iterations, and after the 180th iteration, the solution of system is no more feasible. Fig. 8 and 9 show that even start from a feasible point, no feasible equilibrium can be found if the condition of existence of equilibrium proposed by Lemma 3 is not satisfied.

For the two-OD two-route network, we set the length and saturation flow of two routes of OD1 and OD2 as 1km, 2km, 1km, 2km, and 100 vehicles/h, 350 vehicles/h and 100 vehicles/h, 300 vehicles/h, respectively. The turning matrix is set to be

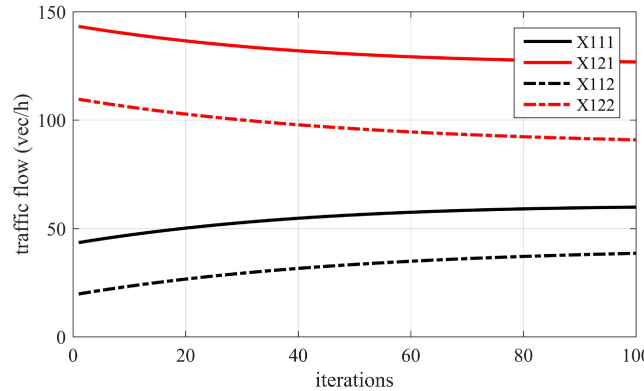

Fig. 13. Time evolution of green-time proportion using novel control policy on two-OD two-route network; the condition of existence of equilibrium is NOT fulfilled.

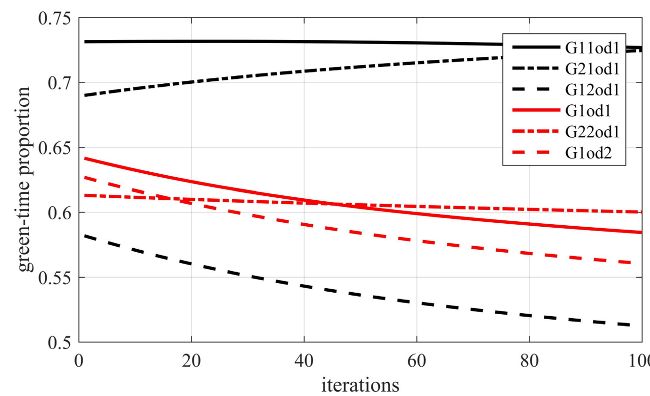

Fig. 14. Time evolution of green-time proportion using novel control policy on two-OD two-route network; the condition of existence of equilibrium is NOT fulfilled.

$$P' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.8 & 0 & 0.2 & 0 \\ 0.64 & 0 & 0.16 & 0.2 \\ 0.04 & 0.8 & 0.16 & 0 \\ 0.64 & 0.64 & 0.136 & 0.16 \\ 0.2 & 0 & 0.8 & 0 \\ 0.16 & 0.2 & 0.64 & 0 \\ 0.16 & 0 & 0.04 & 0.8 \\ 0.136 & 0.16 & 0.064 & 0.64 \end{bmatrix}$$

The free speed of routes of two OD pairs are set to be 50km/h, the total demand of two OD pairs are set to be 186.75km/h and 129.43km/h, respectively. Fig. 11 and Fig. 12 illustrate the time evolution of traffic flow and green-time proportion by using the novel control policy on the two-OD two-route network. Under the condition of equilibrium existence, starting from an arbitrary feasible point, the dynamical system can approach a feasible equilibrium point after approximately 500 iterations. Fig. 13 and Fig. 14 show that the traffic flow and green-time proportion become unfeasible after approximately 100th iteration, that is, no feasible equilibrium point can be approached even with a feasible initial point if the condition of equilibrium existence is not fulfilled.

VI. Conclusions

In this paper, a hybrid traffic assignment model incorporating the flow swap process, green-time proportion swap process and flow divergence was proposed for a general network with multiple OD pairs and multiple routes. We gave two conditions for achieving general network capacity maximization by using P0 control policy. Then, a novel control policy is proposed and its superiority over P0 control policy is proved: by using the proposed control policy, the condition of capacity maximization via intentionally constructing bottleneck delays can be relieved. We gave the condition for existence of Wardrop equilibrium for the dynamical system, which is determined by the sign of flow swap and green-time proportion swap for each element on the boundary of the set $\boldsymbol{D} \times \boldsymbol{F} \cap \boldsymbol{S}$. The condition for uniqueness of Wardrop equilibrium was also given, that is, for a given green-time proportion vector, the travel cost function and stage pressure function must be monotone. We also gave the characteristics of the set of equilibria when green-time proportion vector changes over time, which is useful to determine the set of equilibria. Finally, Lyapunov stability analysis was utilized to prove stability of the dynamical system.

The theoretical results derived in this paper and the proposed control policy can be applied in real scenario to increase network capacity and reduce travel time in an urban network. Specifically, when travel costs on some links go high due to incident, network capacity can be maintained at an equilibrium by applying the proposed control policy.

References

[1] M. Smith, "Traffic signal control and route choice: A new assignment and control model which designs signal timings," *Transp. Res. C, Emerg. Technol.*, vol. 58, pp. 451–473, Sep. 2015. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0968090X1500042X

[2] M. G. H. Bell, F. Kurauchi, S. Perera, and W. Wong, "Investigating transport network vulnerability by capacity weighted spectral analysis," *Transp. Res. B, Methodol.*, vol. 99, pp. 251–266, May 2017. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261516307159

[3] N. S. Abdullah and T. K. Hua, "Using ford-fulkerson algorithm and max flow-min cut theorem to minimize traffic congestion in kota kinabalu, sabah," *J. Inf.*, vol. 2, no. 4, pp. 18–34, Jun. 2017.

[4] M. Smith, R. Liu, and R. Mounce, "Traffic control and route choice: Capacity maximisation and stability," *Transp. Res. B, Methodol.*, vol. 81, pp. 863–885, Nov. 2015. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261515001496

[5] L. Xiao and H. K. Lo, "Combined route choice and adaptive traffic control in a day-to-day dynamical system," *Netw. Spatial Econ.*, vol. 15, no. 3, pp. 697–717, Sep. 2015. doi: 10.1007/s11067-014-9248-4.

[6] T. Le, P. Kovács, N. Walton, H. L. Vu, L. L. H. Andrew, and S. S. P. Hoogendoorn, "Decentralized signal control for urban road networks," *Transp. Res. C, Emerg. Technol.*, vol. 58, pp. 431–450, Sep. 2015. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0968090X14003325

[7] H. Yang, M. G. Bell, and Q. Meng, "Modeling the capacity and level of service of urban transportation networks," *Transp. Res. B, Methodol.*, vol. 34, no. 4, pp. 255–275, May 2000. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261599000247

[8] M. J. Smith, "The existence, uniqueness and stability of traffic equilibria," *Transp. Res. B, Methodol.*, vol. 13, no. 4, pp. 295–304, Dec. 1979. [Online]. Available: http://www.sciencedirect.com/science/article/pii/0191261579900225

[9] M. J. Smith, "Dynamics of route choice and signal control in capacitated networks," *J. Choice Model.*, vol. 4, no. 3, pp. 30–51, 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S1755534513700411

[10] A. Chen and P. Kasikitwiwat, "Modeling capacity flexibility of transportation networks," *Transp. Res. A, Policy Pract.*, vol. 45, no. 2, pp. 105–117, Feb. 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0965856410001576

[11] S.-W. Chiou, "Maximizing reserve capacity for a signalized road network design problem," in *Proc. IEEE Intell. Transp. Syst. Conf.*, Sep. 2006, pp. 1090–1095.

[12] F. Webster, *Traffic Signal Settings* (Road Research Technical Paper). London, U.K.: H.M.S.O, 1958. [Online]. Available: https://books.google.com.au/books?id=c9QOQ4jXK5cC

[13] M. J. Smith, "The stability of a dynamic model of traffic assignment—An application of a method of Lyapunov," *Transp. Sci.*, vol. 18, no. 3, pp. 245–252, Aug. 1984. doi: 10.1287/trsc.18.3.245.

[14] L. Tassiulas and A. Ephremides, "Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks," *IEEE Trans. Autom. Control*, vol. 37, no. 12, pp. 1936–1948, Dec. 1992.

[15] R. Vincent, A. Mitchell, and D. Robertson, *User Guide to TRANSYT Version 8*, Transp. Road Res. Lab., Wokingham, U.K., TRRL Lab., Tech. Rep. 0305-1293 LR 888, 1980. [Online]. Available: https://books.google.com.au/books?id=UuGDGAAACAAJ

[16] Y. M. Nie, "A class of bush-based algorithms for the traffic assignment problem," *Transp. Res. B, Methodol.*, vol. 44, no. 1, pp. 73–89, Jan. 2010. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261509000769

[17] C. B. McGuire, C. B. Winsten, and M. Beckmann, *Studies in the Economics of Transportation*. New Haven, CT, USA: Yale Univ. Press, 1956. [Online]. Available: https://books.google.com.au/books?id=3CVPAAAAMAAJ

[18] R. Mounce and M. Carey, "Route swapping in dynamic traffic networks," *Transp. Res. B, Methodol.*, vol. 45, no. 1, pp. 102–111, Jan. 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261510000809

[19] E. Cascetta, M. Gallo, and B. Montella, "Models and algorithms for the optimization of signal settings on urban networks with stochastic assignment models," *Ann. Oper. Res.*, vol. 144, no. 1, pp. 301–328, Apr. 2006. doi: 10.1007/s10479-006-0008-9.

[20] G. Cantarella and A. Sforza, *Network Design Models and Methods for Urban Traffic Management*. Berlin, Germany: Springer, 1995, pp. 123–153.

[21] S. Wong and H. Yang, "Reserve capacity of a signal-controlled road network," *Transp. Res. B, Methodol.*, vol. 31, no. 5, pp. 397–402, Oct. 1997. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261597000027

[22] Wikipedia Contributors. (2016). *Vertical Queue—Wikipedia, the Free Encyclopedia*. Accessed: Jan. 11, 2019]. [Online]. Available: https://en.wikipedia.org/w/index.php?title=Vertical_queue&oldid=7153444%55

[23] Y. Sheffi, *Urban Transportation Networks: Equilibrium Analysis With Mathematical Programming Methods*. Upper Saddle River, NJ, USA: Prentice-Hall, 1984. [Online]. Available: https://books.google.com.au/books?id=zx1PAAAAMAAJ

[24] X. Wu and H. X. Liu, "A shockwave profile model for traffic flow on congested urban arterials," *Transp. Res. B, Methodol.*, vol. 45, no. 10, pp. 1768–1786, Dec. 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0191261511001135

[25] M. J. Smith, "A descent algorithm for solving monotone variational inequalities and monotone complementarity problems," *J. Optim. Theory Appl.*, vol. 44, no. 3, pp. 485–496, Nov. 1984. doi: 10.1007/BF00935463.

[26] Wikipedia Contributors. (2018). *Critical Point (Mathematics)—Wikipedia, the Free Encyclopedia*. Accessed: Jan. 13, 2019. [Online]. Available: https://en.wikipedia.org/w/index.php?title=Critical_point_(mathematics)&oldid=868851541

Shangbo Wang (S'17–M'18) received the master's and Dr.Ing. degrees from the University of Duisburg-Essen, Germany, in 2007 and 2014, respectively. Prior to joining the University of Technology Sydney as a Research Assistant in 2015, he was with Siemens AG, Munich, Germany, as a Research Engineer, and Continental AG, Lindau, Germany, as a Development Engineer. His research interest includes statistical signal processing, intelligent transportation systems, digital signal processing, wireless localization, and mobile communication.

Changle Li (M'09–SM'16) received the Ph.D. degree in communication and information system from Xidian University, China, in 2005. He conducted his post-doctoral research in Canada and the National Institute of information and Communications Technology, Japan. He has been a Visiting Scholar with the University of Technology Sydney. He is currently a Professor with the State Key Laboratory of Integrated Services Networks, Xidian University. His research interests include intelligent transportation systems, vehicular networks, mobile ad hoc networks, and wireless sensor networks.

Wenwei Yue received the B.E. degree in information engineering from Xidian University, Xi'an, China, in 2015, where he is currently pursuing the Ph.D. degree in communication and information systems. He was also a Visiting Ph.D. Student with the School of Computing and Communications, University of Technology Sydney, from 2017 to 2019. His current research interests include intelligent transportation systems, vehicular network, and big data.

Guoqiang Mao (S'98–M'02–SM'07–F'18) was with the School of Electrical and Information Engineering, University of Sydney. The Center is among the largest university research centers in Australia in wireless communications and networking. In 2014, he joined the University of Technology Sydney as a Professor of wireless networking and the Director of the Center for Real-time Information Networks. He has authored or coauthored about 200 papers in international conferences and journals, which have been cited more than 4 500 times. His research interests include intelligent transport systems, applied graph theory and its applications in telecommunications, the Internet of Things, wireless sensor networks, wireless localization techniques, and network performance analysis. He was a recipient of the Top Editor Award for outstanding contributions to the IEEE TRANSACTIONS ON VEHICULAR TECHNOLOGY, in 2011, 2014, and 2015. He has served as a Chair, a Co-Chair, and a TPC member in a large number of international conferences. He is a Co-Chair of the IEEE Intelligent Transport Systems Society Technical Committee on Communication Networks. He has been an Editor of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS since 2014 and the IEEE TRANSACTIONS ON VEHICULAR TECHNOLOGY since 2010.