Connectivity of Large-Scale CSMA Networks

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Abstract—**Wireless multi-hop networks are being increasingly used in military and civilian applications. Connectivity is a prerequisite in wireless multi-hop networks for providing many network functions. In a wireless network with many concurrent transmissions, signals transmitted at the same time will mutually interfere with each other. In this paper we consider the impact of interference on the connectivity of CSMA networks. Specifically, consider a network with n nodes uniformly and *i.i.d.* on a square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ where a node can only transmit if the sensed power from any other active transmitter is below a threshold, i.e. subject to the carrier-sensing constraint, and the transmission is successful if and only if the SINR is greater than or equal to a predefined threshold. We provide a sufficient condition and a necessary condition, i.e. an upper bound and a lower bound on the transmission power, required for the above network to be asymptotically almost surely (a.a.s.) connected as $n \to \infty$. The two bounds differ by a constant factor only as $n \to \infty$. It is shown that the transmission power only needs to be increased by a constant factor to combat interference and maintain connectivity compared with that considering a unit disk model (UDM) without interference. This result is also in stark contrast with previous results considering the connectivity of ALOHA networks under the SINR model.**

Index Terms—**Connectivity, CSMA, Wireless Network.**

I. INTRODUCTION

Wireless multi-hop networks are being increasingly used in military and civilian applications. Connectivity is a prerequisite in wireless multi-hop networks for providing many network functions (e.g. routing, localization and topology control) [1]–[3]. The scaling behavior of the connectivity property when the network becomes sufficiently large is of particular interest. A wireless multi-hop network is said to be *connected* if and only if (iff) there is at least one (multi-hop) path between any pair of nodes in the network.

Due to the nature of wireless communications, signals transmitted at the same time will mutually interfere with each other. The SINR (signal to interference plus noise ratio) model has been widely used to capture the impact of interference on network connectivity [2], [4], [5]. Under the SINR model, the existence of a directional link between a pair of nodes

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is determined by the strength of the received signal from the desired transmitter, the interference caused by other concurrent transmissions and the background noise. Assume all nodes use the same transmit power P and let $\boldsymbol{x}_k$, $k \in \Gamma$, be the location of node k, where Γ represents the set of indices of all nodes in the network. A node j can successfully receive the transmitted signal from a node i (i.e. node j *is directly connected to* node i) if the SINR at $\boldsymbol{x}_j$, denoted by SINR $(\boldsymbol{x}_i \to \boldsymbol{x}_j)$, is above a prescribed threshold β, i.e.

$$\text{SINR}\,(\boldsymbol{x}_i \to \boldsymbol{x}_j) = \frac{P\ell\,(\boldsymbol{x}_i,\,\boldsymbol{x}_j)}{N_0 + \gamma \sum_{k \in \mathcal{T}_i} P\ell\,(\boldsymbol{x}_k,\,\boldsymbol{x}_j)} \geq \beta \quad (1)$$

where $\mathcal{T}_i \subseteq \Gamma$ denotes the subset of nodes transmitting at the same time as node i and N_0 is the background noise power. The function $\ell\,(\boldsymbol{x}_i,\,\boldsymbol{x}_j)$ is the power attenuation from $\boldsymbol{x}_i$ to $\boldsymbol{x}_j$. The coefficient $0 \leq \gamma \leq 1$ is the inverse of the processing gain of the system and it weighs the impact of interference. In a broadband system using CDMA, γ depends on the orthogonality between codes used during concurrent transmissions and $\gamma < 1$; in a narrow-band system, $\gamma = 1$ [2], [5]. Similarly, node i can receive from node j (i.e. node i is directly connected to node j) iff

$$\text{SINR}\,(\boldsymbol{x}_j \to \boldsymbol{x}_i) = \frac{P\ell\,(\boldsymbol{x}_j,\,\boldsymbol{x}_i)}{N_0 + \gamma \sum_{k \in \mathcal{T}_j} P\ell\,(\boldsymbol{x}_k,\,\boldsymbol{x}_i)} \geq \beta. \quad (2)$$

Therefore node i and node j are directly connected, i.e. a bidirectional link exists between node i and node j, iff both (1) and (2) are satisfied.

Dousse *et al.* [5] use the SINR model to analyze the impact of interference on connectivity from the percolation perspective. They consider a network where all nodes are distributed in $\Re^2$ following a homogeneous Poisson point process with a constant intensity λ and an attenuation function ℓ with bounded support. By letting $\mathcal{T}_j = \Gamma/\{i, j\}$, i.e. all other nodes in the network transmit simultaneously with node i irrespective of their relative locations to $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, it is shown that there exists a very small positive constant γ' such that if $\gamma > \gamma'$ there is no infinite connected component in the network, i.e. the network does not *percolate*. Further, when $\gamma < \gamma'$, there exists $0 < \lambda' < \infty$ such that percolation can occur when $\lambda > \lambda'$. An improved result by the same authors in [6] shows that under the more general conditions that $\lambda > \lambda_c$ and the attenuation function has unbounded support, percolation occurs when $\gamma < \gamma'$. Here λ_c is the critical node density above which the network with $\gamma = 0$ (i.e. the unit disk model (UDM) without interference) percolates [7, p48]. These results suggest that percolation under the SINR model can happen iff γ is sufficiently small. They assume that each node transmits randomly and independently, irrespective

of any nearby transmitter. This corresponds to the ALOHA-type multiple access scheme [2], which however has become obsolete [8].

The more advanced multiple access strategies, e.g. CSMA and CSMA/CD (Carrier Sense Multiple Access with Collision Detection) [9] have become prevailing with widespread adoption. The general idea of CSMA schemes is that nearby nodes will not be scheduled to transmit simultaneously, i.e., a minimum separation distance is imposed among concurrent transmitters**.** Therefore, it is natural to expect that CSMA could improve the performance of ALOHA schemes by alleviating interference, particularly under heavy traffic. On the other hand, this distance constraint leads to a spatial correlation problem which means that the location of a transmitter is dependent on the location of other concurrent transmitters. Therefore, even if all nodes are initially distributed following a Poisson point process (PPP), the set of concurrent transmitters cannot be obtained by independent thinning of the PPP. Thus, the set of concurrent transmitters no longer forms a PPP but a more complicated point process. *Matérn hard-core point process* are widely used to model the set of concurrent transmitters [10]–[12]. However, distribution of such hard-core process is difficult to analyze and a closed-form expression is yet to be obtained [10]–[14]. In this paper, we use an entirely different approach. Particularly by investigating the bounds on interference, instead of an accurate characterization of interference distribution, we are able to avoid the above mentioned difficulty in finding the accurate distribution of concurrent transmitters and the associated interference.

In this paper, we analyze the connectivity of wireless CSMA networks under the SINR model. Specifically, we consider a network with n nodes uniformly *i.i.d.* on a square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ and each node is capable of performing carrier-sensing operation. A pair of nodes are directly connected iff both (1) and (2) are satisfied. Further, the attenuation function assumes a power-law form, the same model considered in [5], [6]. The contributions of this paper are:

1) We show that the interference experienced by any receiver in the network is upper bounded. Based on this result, we further show that for an arbitrarily chosen SINR threshold, there exists a transmission range R_0 such that a pair of nodes are directly connected if their Euclidean distance is smaller than or equal to R_0. On that basis, we derive a sufficient condition, i.e. an upper bound on the transmission power, for the CSMA network to be *a.a.s.* connected under the SINR model as $n \to \infty$. An event ξ_n depending on n is said to occur *a.a.s.* if its probability tends to 1 as $n \to \infty$.

2) We provide a necessary condition, i.e. a lower bound on the transmission power, for the CSMA network to be *a.a.s.* connected. The lower bound is a tight bound and differs from the upper bound by a constant factor only.

3) We show that the transmission power only needs to be increased by a constant factor to combat interference and maintain connectivity compared with that considering a UDM without interference. This result is in stark contrast with previous results considering the connectivity

of ALOHA networks [5], [6] under the SINR model which shows that connectivity is much harder to achieve in the presence of interference and is impossible in a narrow band system where $\gamma = 1$.

The remainder of this paper is organized as follows: Section II reviews related work; Section III defines network and connection models. In Section IV we first derive an upper bound on the interference in CSMA networks. Based on the upper bound, a sufficient condition for connectivity is obtained; Section V investigates a necessary condition for a connected CSMA network; finally Section VI concludes the paper and discusses future work.

II. RELATED WORK

The literature is rich in studying connectivity using the well-known *random geometric graph* and the UDM, which is usually obtained by randomly and uniformly distributing n nodes in a given area and connecting any two nodes iff their Euclidean distance is smaller than or equal to a certain threshold $r(n)$, known as the *transmission range*. This model corresponds to a special case of the SINR model in (1), i.e. when $\gamma = 0$ (perfect orthogonality, no interference). Significant outcomes have been achieved for both asymptotically infinite n [1], [15] and for finite n [16]–[18]. Particularly, Penrose [15] and Gupta and Kumar [1] prove that under the UDM and in a disk of unit area, the above network with a transmission range of $r(n) = \sqrt{\frac{\log n + c(n)}{\pi n}}$ is *a.a.s.* connected as $n \to \infty$ iff $c(n) \to \infty$. Most of the results for finite n are empirical results.

The work [3], [19]–[21] investigate the necessary condition for the above network to be *a.a.s.* connected under the more realistic *log-normal connection model*, where two nodes are directly connected if the received power at one node from the other node, whose attenuation follows the log-normal model [9], is greater than a given threshold. These results however rely on the assumption that the node isolation events are independent, which is yet to be proved.

Despite the significant impact of interference caused by concurrent transmissions on connectivity, limited work exists on analyzing connectivity under the SINR model. In [22], [23], the authors study connectivity from the perspective of channel assignment. Specifically, channel/time slots are assigned to each link for all active links to be simultaneously transmitting while satisfying the SINR requirement. The two papers [5], [6] discussed in Section I study the impact of interference on the connectivity of ALOHA networks from the percolation perspective.

Mao *et al.* [24], [25] study the connectivity problem under a generic *random connection model*, viz. two nodes separated by a Euclidean distance x are directly connected with probability $g(x)$, where $g : [0, \infty) \to [0, 1]$ satisfies the properties of integral boundedness, rotational invariance and non-increasing monotonicity [7], independent of the event that another pair of nodes are directly connected. The authors establish the requirements for an *a.a.s.* connected network.

A major difficulty in moving to the SINR model is that under the unit disk model or the random connection model, connections are assumed to be independent, i.e. the event that a pair of nodes are directly connected and the event that another distinct pair of nodes are directly connected are independent. This independence assumption on connections is critical in the analysis of connectivity under these two models. In the SINR model however, due to the presence of interference, the existence of a direct connection between a pair of nodes depends on both the location and the activities of other nodes in the network.

Some other work exists on modeling the point process formed by concurrent transmitters. The Matérn hard-core point process are widely used to model the set of concurrent transmitters in a CSMA network [10]–[13]. However, such hard-core process are difficult to analyze. Consequently, some work [12], [26], [27] uses PPP to approximate the distribution of concurrent transmitters. A recent work [14] compared the mean interference in hard-core process and in the corresponding PPP approximation.

III. NETWORK MODELS

We consider a network with n nodes uniformly and *i.i.d.* on a square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$, i.e. the so-called *extended network model* [2], where the network size scales with the network area while the node density is fixed. All nodes use the same transmission power P and there is always a packet at a node waiting to be transmitted. The later assumption allows us to focus on the network property without being disturbed by other factors, e.g. traffic distribution.

A. Attenuation and interference

We consider that the attenuation function ℓ in (1) and (2) only depends on Euclidean distance and is a power-law function [5], [6] $\ell(s) = s^{-\alpha}$ where s represents the Euclidean distance between a pair of nodes and α is the path-loss exponent, which typically varies from 2 to 6 [9, p139]. In this paper we assume $\alpha > 2$. Note that when $\alpha \leq 2$, the interference experienced by a receiver in the CSMA network investigated can not be bounded by a constant. The above assumptions on ℓ are widely used [2], [5], [12] and supported by measurement studies [9]. As commonly done in the connectivity analysis [1], [5]–[7], [15], the impact of small-scale fading is ignored and only bidirectional communication links are considered. Further, since in dense sensor networks and cellular networks the background noise is typically negligibly small [2], [12], we ignore the background noise N_0 in (1) and (2). In addition, we consider that all nodes use the same channel, i.e. $\gamma = 1$, which corresponds to a narrow-band system [2], [5].

B. Carrier-sensing

In CSMA networks, two nodes located at $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ can respectively transmit simultaneously iff they can not detect each other's transmission, i.e. both $P\ell(\boldsymbol{x}_i, \boldsymbol{x}_j)$ and $P\ell(\boldsymbol{x}_j, \boldsymbol{x}_i)$

in (1) and (2) are below a certain detection threshold P_{th}. From the power-law path loss, the carrier-sensing range R_c, which determines the *minimum* Euclidean distance between two concurrent transmitters, is given by

$$R_c = (P/P_{\text{th}})^{1/\alpha} \tag{3}$$

One may alternatively consider a scenario where a node transmits when the aggregated interference is below P_{th}, which forms a trivial extension of the scenario considered in this paper.

IV. A SUFFICIENT CONDITION FOR ASYMPTOTICALLY ALMOST SURELY CONNECTIVITY

A major challenge in connectivity analysis under the SINR model is that the existence of a direct connection between a pair of nodes depends on both the locations and activities of other nodes in the network, i.e. connections are correlated. In this paper, we resort to a coupling approach to handle the connection correlations. The main idea of coupling technique is to build the connection between a more complicated model and a simpler model with established results such that if a property, e.g. connectivity, is true in the simpler model, it will also be true in the more complicated model. Therefore the property of the more complicated model can be studied by studying the simpler counterpart.

Specifically, we first establish an upper bound on the interference experienced by any receiver in CSMA networks. On that basis, we show that for an arbitrarily chosen SINR threshold, there exists a transmission range R_0 such that a pair of nodes are directly connected if their Euclidean distance is smaller than or equal to R_0. Then we can use existing results on connectivity under the UDM to analyze connectivity under the SINR model.

A. An upper bound on interference and the associated transmission range

The following theorem provides an upper bound on the interference.

Theorem 1. *Consider a CSMA network with nodes distributed arbitrarily on a finite area in $\Re^2$. Denote by r_0 the Euclidean distance between a receiver and its nearest transmitter in the network, which is also the intended transmitter for the receiver. When $r_0 < R_c$, the maximum interference experienced by the receiver is smaller than or equal to $N(r_0) = N_1(r_0) + N_2$, where*

$$N_1(r_0) = \frac{4P\left(\frac{5\sqrt{3}}{4}R_c - r_0\right)^{1-\alpha}\left(\frac{\sqrt{3}}{4}(3\alpha-1)R_c - r_0\right)}{R_c^2(\alpha-1)(\alpha-2)}$$
$$+ \frac{3P}{(R_c - r_0)^\alpha} + \frac{3P}{(\sqrt{3}R_c - r_0)^\alpha} + \frac{3P\left(\frac{3}{2}R_c - r_0\right)^{1-\alpha}}{(\alpha-1)R_c} \tag{4}$$

$$N_2 = \frac{3P}{R_c^\alpha} + \frac{3P(\frac{3}{2})^{1-\alpha}}{(\alpha-1)R_c^\alpha} + \frac{3P}{(\sqrt{3}R_c)^\alpha}$$
$$+ \frac{3P\left(\frac{5}{4}\right)^{1-\alpha}(3\alpha-1)}{(\alpha-1)(\alpha-2)(\sqrt{3}R_c)^\alpha} \tag{5}$$

Proof: See Appendix I. ∎

Remark 2. The upper bound in Theorem 1 is valid for any node distribution. For a sparse network or a network where nodes are placed in a coordinated or planned manner, replacing R_c with the minimum distance among concurrent transmitters, Theorem 1 can be extended to be applicable.

Remark 3. The assumption that $r_0 < R_c$ is valid in most wireless systems which not only require the SINR to be above a threshold and also require the received signal to be of sufficiently good quality. However Theorem 1 does not critically depend on the assumption. For $r_0 \geq R_c$, so long as there exists a positive integer c such that $r_0 < cR_c$ the upper bound can be revised to accommodate the situation by changing the range of the summation in (20) (in Appendix I) from $[3, \infty]$ and $[2, \infty]$ to $[c + 2, \infty]$ and $[c + 1, \infty]$ respectively and revising the results accordingly.

The following result can be obtained as a ready consequence of Theorem 1.

Corollary 4. *Under the same settings as in Theorem 1, assume that the SINR threshold in (1) and (2) is β. There exists a transmission range $R_0 < R_c$ such that a pair of nodes are directly connected if their Euclidean distance is smaller than or equal to R_0, given implicitly by*

$$PR_0^{-\alpha}/N(R_0) = \beta \tag{6}$$

Proof: Theorem 1 establishes that the interference experienced by a receiver z at r_0 from its transmitter w, denoted by $I(r_0)$ is upper bounded by $N(r_0)$. Note that, for $r_0 < R_c$, $N(r_0)$ is increasing with r_0 and $Pr_0^{-\alpha}$ is decreasing with r_0. Therefore, using (6) the SINR of a receiver at $r_0 \leq R_0$ from its transmitter, denoted by SINR(r_0), satisfies SINR$(r_0) = \frac{Pr_0^{-\alpha}}{I(r_0)} \geq \frac{Pr_0^{-\alpha}}{N(r_0)} \geq \beta$.

By symmetry, when the transmission occurs in the opposite direction, i.e. from z to w, the interference generated by the set of nodes that are transmitting at the same time as z is also upper bounded by $N(r_0)$. Therefore the SINR at w is also greater than or equal to β.

Finally the existence of a (unique) solution to (6) can be proved by noting that $\frac{Pr_0^{-\alpha}}{N(r_0)} \to \infty$ as $r_0 \to 0$, $\frac{Pr_0^{-\alpha}}{N(r_0)} \to 0$ as $r_0 \to R_c^-$ and that $\frac{Pr_0^{-\alpha}}{N(r_0)}$ is monotonically decreasing with r_0. ∎

Corollary 4 relates R_0 to P and allows the computation of R_0 given P and the converse. A more convenient way to study the relation between P and R_0 is by noting that $P = P_{\text{th}} R_c^{\alpha}$ and considering R_0 as a function of R_c. Using (4), (5) and letting $\frac{R_c}{R_0} = x$, (6) can be rewritten as

$$
\begin{aligned}
\frac{1}{\beta} &= \frac{4\left(\frac{5\sqrt{3}}{4}x - 1\right)^{1-\alpha}\left(\frac{\sqrt{3}}{4}(3\alpha - 1)x - 1\right)}{x^2(\alpha - 1)(\alpha - 2)} \frac{3}{(x - 1)^{\alpha}} \\
&+ \frac{3}{\left(\sqrt{3}x - 1\right)^{\alpha}} + \frac{3\left(\frac{3}{2}x - 1\right)^{1-\alpha}}{(\alpha - 1)x} + \frac{3\left(\frac{3}{2}\right)^{1-\alpha}}{x^{\alpha}(\alpha - 1)} \\
&+ \frac{3}{x^{\alpha}} + \frac{3}{\left(\sqrt{3}x\right)^{\alpha}} + \frac{3\left(\frac{5}{4}\right)^{1-\alpha}(3\alpha - 1)}{(\alpha - 1)(\alpha - 2)\left(\sqrt{3}x\right)^{\alpha}}
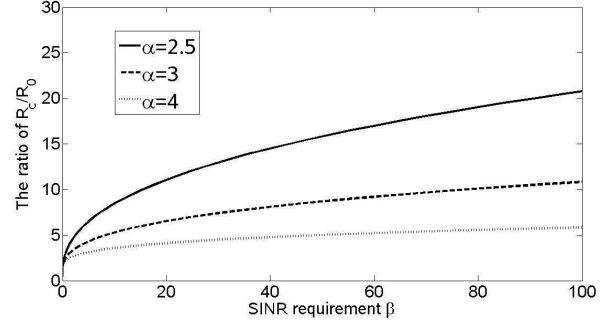\end{aligned} \tag{7}
$$

Figure 1. Variation of the ratio $\frac{R_c}{R_0}$ with the SINR requirement β when the path loss exponent α equals to 2.5, 3, 4, respectively.

Figure 1 shows the ratio $\frac{R_c}{R_0}$ as a function of β. Different curves represent different choices of the path loss exponent α. For instance, when $\beta = 10$ and $\alpha = 4$, we have $\frac{R_c}{R_0} = 3.6$.

B. A sufficient condition for connectivity

Based on the transmission range R_0 derived in Corollary 4, we obtain another main result:

Theorem 5. *Consider a CSMA network with a total of n nodes i.i.d. on a square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ following a uniform distribution. A pair of nodes are directly connected iff both (1) and (2) ($\gamma = 1$ and $N_0 = 0$ in (1) and (2)) are satisfied. As $n \to \infty$, the above network is a.a.s. connected if the transmission power*

$$P = P_{th} b_1^{\alpha}\left(\log n + c(n)\right)^{\frac{\alpha}{2}}, \tag{8}$$

where $b_1 = b'/\sqrt{\pi}$, $c(n) = o(\log n)$ and $c(n) \to \infty$ as $n \to \infty$ and $\infty > b' > 1$ is the solution to (7) (By $f(x) = o(g(x))$, we mean that $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.).

Proof: The results in [1], [15] show that, for a network with a total of n nodes uniformly *i.i.d.* on a $\sqrt{n} \times \sqrt{n}$ square and a pair of nodes are directly connected iff their Euclidean distance is smaller than or equal to a given threshold $r(n)$ (i.e., UDM), the network is *a.a.s.* connected as $n \to \infty$ iff $r(n) = \sqrt{\frac{\log n + c(n)}{\pi}}$ where $c(n) \to \infty$ as $n \to \infty$. Using this result, (7) (letting $b' = \frac{R_c}{R_0}$), Corollary 4 and Theorem 1, the result in the theorem follows. ∎

The implication of Theorem 5 is that in CSMA networks, since the interference is bounded above by a constant almost surely as shown in Theorem 1, to meet an arbitrarily high (albeit constant with the increase in n) β, the power only needs to be increased by a constant factor compared with that in the unit disk model to maintain the same set of connections. This result is in contrast to the ALOHA networks considered in [5], [6] in which percolation only occurs for a sufficiently small γ.

V. A NECESSARY CONDITION FOR ASYMPTOTICALLY ALMOST SURELY CONNECTIVITY

Section IV derives a sufficient condition for a connected CSMA network as $n \to \infty$ in the presence of interference.

A logical question arises: what is the necessary condition for the same CSMA network to be connected as $n \to \infty$.

In a CSMA network, any set of nodes can transmit simultaneously as long as the carrier-sensing constraints are satisfied. Further, in a large-scale network, scheduling is often performed in a distributed manner. In the absence of accurate global knowledge on which particular set of nodes are simultaneously transmitting at a particular time instant, it is natural that a node sets its transmission power to be above *the minimum transmission power* required for a network to be connected under *any* scheduling algorithm (It is trivial to show that, see also the proof of Lemma 6, when the transmission power increases, connectivity will also improve). Denote that minimum power by P'_Ω where Ω represents the set of all scheduling algorithms satisfying the carrier-sensing constraints. In this section, we investigate P'_Ω, i.e. a necessary condition required for connectivity as $n \to \infty$. This is done by analyzing the transmission power required for the above network to have no isolated node which is a necessary condition for having a connected network. The following lemma is required for the analysis of P'_Ω:

Lemma 6. *Denote by P_Ω (respectively, P_ω) the minimum transmission power required for the network to have no isolated node under any scheduling (respectively, under a particular scheduling ω). We have $P'_\Omega \geq P_\Omega = \max_{\omega \in \Omega} P_\omega$.*

Proof: We prove the lemma by showing that the minimum transmission power required for the network to have no isolated node under *any* scheduling has to be greater than or equal to the minimum transmission power required for the same network to have no isolated node under a particular scheduling.

Define a set of nodes that can simultaneously transmit while satisfying the carrier-sensing constraints as an *independent set*. Obviously the independent set depends on the transmission power of nodes. As the transmission power decreases, other things being equal, R_c will decrease and the number of nodes that can simultaneously transmit will increase or remain the same.

Denote by ϕ' a set of nodes that are scheduled to transmit simultaneously in the CSMA network. It follows that ϕ' must be an independent set. *Given* ϕ', a node $v \in \phi'$ is *isolated* if there is no node in the network that can successfully receive from it when the nodes in ϕ' are simultaneously transmitting. Further, as explained in the last paragraph, the independent set depends on the transmission power. When the transmission power is decreased from P_1 to P_2, where $P_2 \leq P_1$, if ϕ' is an independent set at power level P_1, it will also be an independent set at power level P_2. Based on the above observation and using (1) and (2), a decrease in the transmission power will cause a decrease in the SINR, it readily follows that if a node $v \in \phi'$ is isolated at power level P_1 when the set of active transmitters is ϕ', it will also be isolated at power level P_2 when the set of active transmitters is ϕ'. For any transmission power less than $P_\Omega = \max_{\omega \in \Omega} P_\omega$, there exists a scheduling that will result the network to have an isolated node at that power level. Therefore, P_Ω has to satisfy
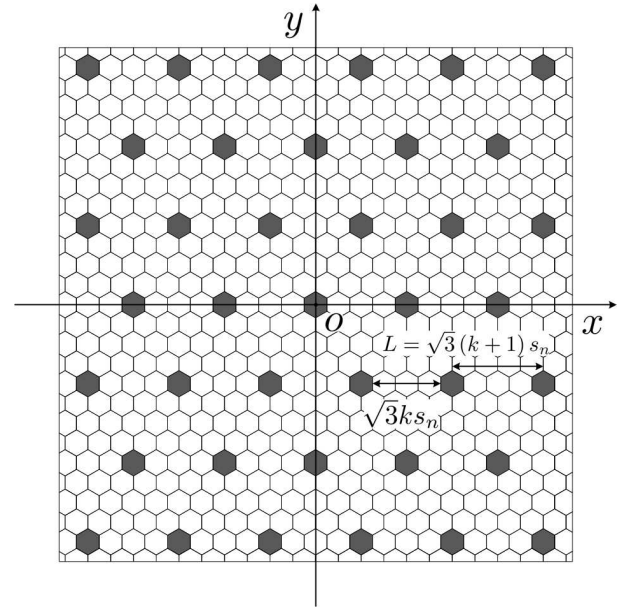

Figure 2. An illustration of the hexagonal partition of the network area. The shaded hexagons represent simultaneously active hexagons, where $k = 3$.

$P_\Omega = \max_{\omega \in \Omega} P_\omega$. ∎

Remark 7. As an easy consequence of Lemma 6, the probability that a CSMA network has no isolated node is a non-increasing function of the transmission power.

Now the task becomes constructing a particular scheduling which gives as large P_ω as possible, i.e. a tight lower bound on P'_Ω. Next we construct such scheduling ω heuristically.

A. Construction of scheduling algorithm ω

Obviously, ω needs to satisfy the constraint on the minimum separation distance between concurrent transmitters imposed by the carrier-sensing requirement. Meanwhile, ω needs to schedule as many concurrent transmissions as possible to maximize interference, hence P_ω.

We start with a lemma that is required for the construction of ω.

Lemma 8. *Partition the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ into non-overlapping hexagons of equal side length s_n such that the origin o coincides with the centre of a hexagon and two diagonal vertices of this hexagon, whose Euclidean distance is $2s_n$, are located on y axis, as shown in Figure 2. We call a hexagon an* interior *hexagon if it is entirely contained in the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$. When $s_n = \sqrt{(2\log n)/5}$, a.a.s. each interior hexagon is occupied by at least one node as $n \to \infty$.*

Proof: Because nodes are uniformly *i.i.d.*, the probability that an arbitrary interior hexagon is empty is $\left(1 - \frac{3\sqrt{3}s_n^2}{2n}\right)^n$. Let ξ_i be the event that an interior hexagon i is empty, where $i \in \Xi$ and Ξ denotes the set of indices of all interior hexagons. There are at most $\frac{2n}{3\sqrt{3}s_n^2}$ interior hexagons.

Denote by A_n the event that there is at least one empty interior hexagon in $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$. It follows that $\Pr(A_n) = \Pr(\cup_{i \in \Xi} \xi_i)$. Using union bound, we have $\Pr(\cup_{i \in \Xi} \xi_i) \leq \sum_{i \in \Xi} \Pr(\xi_i) \leq \frac{2n\left(1 - \frac{3\sqrt{3}s_n^2}{2n}\right)^n}{3\sqrt{3}s_n^2}$. Using the fact that $1 - x \leq \exp(-x)$ and $s_n = \sqrt{\frac{2\log n}{5}}$, we have $\lim_{n \to \infty} \Pr(A_n) \leq \lim_{n \to \infty} \frac{2ne^{-\frac{3\sqrt{3}s_n^2}{2}}}{3\sqrt{3}s_n^2} = \lim_{n \to \infty} \frac{5n}{3\sqrt{3}n^{\frac{3\sqrt{3}}{5}}\log n} = 0$ which completes the proof. ∎

Hereinafter, we declare a hexagon to be *active* if there is a node transmitting in it. We consider a scheduling ω that uses the hexagons as the basic unit for scheduling. Due to the minimum separation distance constraint, any two simultaneously active hexagons should be separated by a minimum Euclidean distance (depending on the carrier-sensing range given in (3)). Let k be an integer and represent the minimum number of inactive hexagons between two closest simultaneously active hexagons (see Figure 2). Any two nodes inside the two active hexagons are separated by a Euclidean distance of at least $\sqrt{3}ks_n$. With a bit twist of terminology, we further define a *maximal independent set* for scheduling to be the set of hexagons that a) includes as many hexagons as possible; and b) closest hexagons in the set are separated by exactly k adjacent hexagons. Figure 2 illustrates such a maximal independent set with $k = 3$.

We define ω such that only hexagons belonging to the same maximal independent set can be active at the same time. No nodes in the same hexagon can be scheduled to transmit simultaneously. (Note that if a hexagon intersecting the border of $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ has node(s) in it, it is also included into the maximal independent set and its node(s) are treated in the same way as other nodes in interior hexagons.) As a consequence of the CSMA constraint and the definition of k,

$$\sqrt{3}ks_n \geq R_c \geq \sqrt{3}(k-1)s_n \tag{9}$$

B. Probability of having no isolated node

In this subsection, we derive a lower bound on P_ω for ω defined in the previous subsection. This is done by analyzing the event that the network has no isolated node under ω. The following theorem summarizes another major outcome of the paper:

Theorem 9. *Under the same setting in Theorem 5 and the scheduling algorithm ω, a necessary condition on P_ω for the CSMA network to have no isolated node a.a.s. as $n \to \infty$ is*

$$P_\omega \geq P_{th}b_2^\alpha (\log n)^{\frac{\alpha}{2}} \tag{10}$$

where $b_2 = \sqrt{6/5}(b-1)$ and b is the smallest integer satisfying the inequality: $\frac{2\left(\sqrt{3}(b+1)+1\right)^{1-\alpha}\left(\sqrt{3}(\alpha-1)(b+1)+1\right)}{(b+1)^2(\alpha-1)(\alpha-2)} \leq \frac{1}{\beta}\left(\frac{2\pi}{5}\right)^{\frac{\alpha}{2}}$.

Proof: The main strategy used is to couple the network under the SINR model with the associated network under UDM. Then, an upper bound on the probability of having no

isolated node in the network under the SINR model is obtained by using existing results for UDM.

Denote the Euclidean distance between the centers of two closest hexagons in a maximal independent set by $L = \sqrt{3}(k+1)s_n$. See Figure 2 for an illustration. Divide the hexagons belonging to the same maximal independent set as a hexagon h_i into tiers of increasing Euclidean distance from the centre of h_i using a similar strategy as that in the proof of Theorem 1. The mth tier of h_i has at most $6m$ hexagons. Further, we declare that the mth tier of h_i is *complete in a given area* if all the $6m$ hexagons are entirely enclosed in this given area. Denote by C_A a square $\left[-\frac{\sqrt{cn}}{2}, \frac{\sqrt{cn}}{2}\right]^2$ ($0 < c < 1$ and the exact value of c will be decided later in this paragraph). The hexagon containing the origin o has a number of $t = \left\lfloor \frac{\frac{c\sqrt{n}}{2} - \frac{\sqrt{3}s_n}{2}}{L} \right\rfloor$ complete tiers in C_A. As c increases, t increases as well. For the hexagons located in C_A but near the border of C_A, the number of complete tiers in the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ decreases with an increase in c. We choose the value of c such that each hexagon inside C_A has at least t complete tiers in the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$, and the value of t is maximized. Let C_A' be the union of hexagons entirely contained in C_A. With a little bit abuse of terminology, we use C_A (C_A') to denote both the area itself and the size of the area. We can obtain $\lim_{n \to \infty} \frac{C_A'}{C_A} = 1$.

Consider an arbitrarily node i transmitting inside a hexagon h_i in C_A'. If there is no node that can receive from it, then node i is isolated. Let I_{min} be the *minimum* interference that could possibly be experienced by a potential receiver of node i under ω. Note that the Euclidean distance between the transmitter inside a hexagon in the mth tier of h_i and the centre of hexagon h_i is less than $mL + s_n$ (see Figure 2). Using Lemma 12 gives

$$
\begin{aligned}
I_{min} &\geq \sum_{m=1}^{t} 6m(mL + s_n)^{-\alpha}P \\
&= 6Ps_n^{-\alpha}\sum_{m=1}^{t} m\left(\sqrt{3}m(k+1)+1\right)^{-\alpha} \\
&= 6Ps_n^{-\alpha}\int_1^t \lfloor x \rfloor \left(\sqrt{3}\lfloor x \rfloor(k+1)+1\right)^{-\alpha}dx \quad (11) \\
&\geq 6Ps_n^{-\alpha}\int_1^t x\left(\sqrt{3}x(k+1)+1\right)^{-\alpha}dx \quad (12)
\end{aligned}
$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x. (12) is obtained due to the fact that $x\left(\sqrt{3}x(k+1)+1\right)^{-\alpha}$ is a decreasing function when $x > \frac{1}{\sqrt{3}(k+1)(\alpha-1)}$ and $\sqrt{3}(k+1)(\alpha-1) > 1$ for $\alpha > 2$ and $k \geq 1$. Therefore $x\left(\sqrt{3}x(k+1)+1\right)^{-\alpha}$ is a decreasing function when $x > 1$. Further, noting that $\lim_{n \to \infty} t = \lim_{n \to \infty}\left\lfloor \frac{\frac{\sqrt{cn}}{2} - \frac{\sqrt{3}s_n}{2}}{L} \right\rfloor = \infty$, it follows that

$$
\begin{aligned}
&\lim_{n \to \infty} 6\int_1^t x\left(\sqrt{3}x(k+1)+1\right)^{-\alpha}dx \\
&= \frac{2\left(\sqrt{3}(k+1)+1\right)^{1-\alpha}\left(\sqrt{3}(\alpha-1)(k+1)+1\right)}{(k+1)^2(\alpha-1)(\alpha-2)} \\
&\triangleq f(k)
\end{aligned}
$$

The above equation implies that for an arbitrarily small positive constant ε, there exists a positive integer n_ε such that when $n \geq n_\varepsilon$

$$\text{RHS of (12)} \geq Ps_n^{-\alpha}\left(f(k) - \varepsilon\right) \triangleq J_n \qquad (13)$$

Let d be the Euclidean distance between node i and its receiver. By (1), (2), it follows that *only* when $\frac{Pd^{-\alpha}}{J_n} \geq \beta$, the transmission from node i to its receiver could *possibly* be successful. In other words, if there is no node within a Euclidean distance of $R = \left(\beta J_n/P\right)^{-\frac{1}{\alpha}}$ to node i, then it is isolated.

Denote by M and M^{SINR} the (random) number of isolated nodes in the CSMA network in the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ and in $C'_A \subset \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ respectively. Denote by M^{UDM} the (random) number of isolated nodes in an area $C'_A \subset \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ in a network with a total of n nodes uniformly i.i.d. on the square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ under UDM with the transmission range R. Based on the discussion in the last paragraph and using the coupling technique [7], it can be shown that $\Pr\left(M \geq 1\right) \geq \Pr\left(M^{\text{SINR}} \geq 1\right) \geq \Pr\left(M^{\text{UDM}} \geq 1\right)$. Consequently,

$$\Pr\left(M = 0\right) \leq \Pr\left(M^{\text{UDM}} = 0\right) \qquad (14)$$

It remains to find the value of $\Pr\left(M^{\text{UDM}} = 0\right)$. We first consider a network with a total of n nodes distributed on a square $\left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$ under UDM with a transmission range $r(n)$. It is well-known that when the average node degree in the above network equals to $\log n + \zeta(n)$ and $\lim_{n\to\infty} \zeta(n) = \zeta$ where ζ is a constant ($\zeta = \infty$ is allowed), the probability that there is no isolated node in the above network asymptotically converges to $e^{-e^{-\zeta}}$ as $n \to \infty$ [7], [28], [29]. Further, it was shown in [30] that boundary effect has an asymptotically vanishingly impact on the number of isolated nodes. Let Z be a random integer representing the number of nodes located inside $C_A \subset \left[-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}\right]^2$. $\text{E}(Z) = cn$ and $\text{Var}(Z) = cn(1-c)$. Let $M^{r(n)}$ be the number of isolated nodes within an area C_A in the above network with a transmission range $r(n)$. Based on the above results, conditioned on that $Z = cn$ we have (here we have omitted some trivial discussions involving the situation that cn is not an integer)

$$\lim_{n\to\infty} \Pr\left(M^{r(n)} = 0 \,\Big|\, Z = cn\right) = e^{-ce^{-\zeta}} \qquad (15)$$

Using Chebyshev's inequality, for $0 < \delta < \frac{1}{2}$, we obtain that

$$\lim_{n\to\infty} \Pr\left(|Z - cn| \geq (cn)^{\frac{1}{2}+\delta}\right) \leq \lim_{n\to\infty} \frac{\text{Var}(Z)}{\left((cn)^{\frac{1}{2}+\delta}\right)^2} = 0 \qquad (16)$$

Let $f(n) = (cn)^{\frac{1}{2}+\delta}$. Using the following two equations: $\log\left(n + f(n)\right) + \zeta(n) = \log n + \log\left(1 + \frac{f(n)}{n}\right) + \zeta(n)$ and $\lim_{n\to\infty} \log\left(1 + \frac{f(n)}{n}\right) + \zeta(n) = \lim_{n\to\infty} \zeta(n) = \zeta$ and (15), it can be shown that $\lim_{n\to\infty} \Pr\left(M^{r(n)} = 0 \,\Big|\, Z = cn + f(n)\right) = e^{-ce^{-\zeta}}$.
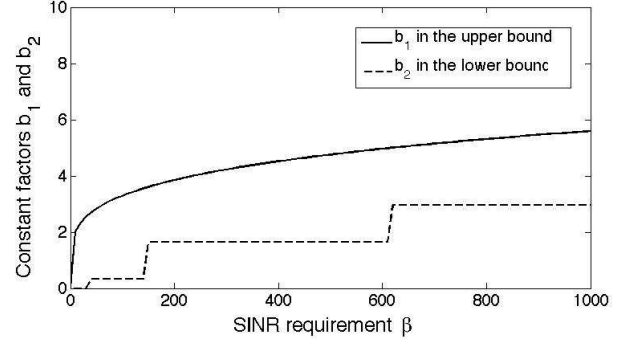

Figure 3. A plot of the two constant factors b_1 and b_2 in the upper bound (8) and in the lower bound (18) when $\alpha = 4$.

Hence, for any integer m satisfying $-f(n) \leq m \leq f(n)$, $\lim_{n\to\infty} \Pr\left(M^{r(n)} = 0 \,\Big|\, Z = cn + m\right) = e^{-ce^{-\zeta}}$. This equation, together with (16), allows us to conclude that when $r(n) = \sqrt{\frac{\log n + \zeta(n)}{\pi}}$

$$\lim_{n\to\infty} \Pr\left(M^{r(n)} = 0\right) = e^{-ce^{-\zeta}} \qquad (17)$$

As a result of (14), a necessary condition for $\lim_{n\to\infty} \Pr\left(M = 0\right) = 1$ is that $\lim_{n\to\infty} \Pr\left(M^{\text{UDM}} = 0\right) = 1$. Using the fact that $\lim_{n\to\infty} \frac{C'_A}{C_A} = 1$ and (17), it follows that a necessary condition for the network under the SINR model to *a.a.s.* have no isolated node is that $R \geq \sqrt{\frac{\log n + \zeta(n)}{\pi}}$ and $\zeta(n) \to \infty$ as $n \to \infty$. As denoted $R = \left(\beta J_n/P\right)^{-\frac{1}{\alpha}}$, together with the value of J_n in (13) and the value of s_n in Lemma 8, we obtain that $f(k) \leq \frac{1}{\beta}\left(\frac{2\pi}{5}\frac{\log n}{\log n + \zeta(n)}\right)^{\frac{\alpha}{2}} + \varepsilon$. Letting $n \to \infty$ and then $\varepsilon \to 0$ in the above inequality yields $f(k) \leq \frac{1}{\beta}\left(\frac{2\pi}{5}\right)^{\frac{\alpha}{2}}$. Based on the above equation, together with (3) and (9), Theorem 9 results. ∎

The following corollary is obtained as a ready consequence of Theorem 9 and Lemma (6).

Corollary 10. *A necessary condition required for CSMA networks to be a.a.s. connected as $n \to \infty$ under any scheduling algorithm, i.e. a lower bound on P'_Ω, is given by*

$$P'_\Omega \geq P_{\text{th}} b_2^\alpha \left(\log n\right)^{\frac{\alpha}{2}} \qquad (18)$$

Comparing the lower bound on P'_Ω in (18) with the upper bound in (8) and noting that $c(n) = o(\log n)$, it can be shown that, given an arbitrary β, the two bounds differ by a constant factor only as $n \to \infty$. Figure 3 shows the a plot of the two constant factors, viz. b_1 and b_2, in (8) and in (18) respectively as a function of β when $\alpha = 4$. The curve representing b_2 is a step function due to the granularity caused by the integer k in the scheduling algorithm ω.

VI. Conclusion and Future Work

In this paper, we studied the connectivity of wireless CSMA networks considering the impact of interference. We showed

that, different from an ALOHA network, the aggregate interference experienced by any receiver in CSMA networks is upper bounded even when the coefficient γ in (1) and (2) equals to 1.

An upper bound and a lower bound were obtained on the critical transmission power required for having an *a.a.s.* connected CSMA network. The two bounds are tight and differ by a constant factor only. The results suggested that any pair of nodes can be connected for an arbitrarily high SINR requirement so long as the carrier-sensing capability is available. Compared with that considering UDM without interference, the transmission power only needs to be increased by a constant factor to combat interference and maintain connectivity. This is a optimistic result compared with previous results on the connectivity of ALOHA networks under the SINR model.

The gap between the two bounds can be further narrowed by considering more complicated geometric shapes than hexagons. However such improvement is possibly of minor importance. The implication of the results in this paper is that there exists a spatial and temporal scheduling algorithm in a large scale CSMA network that allows as many as possible concurrent transmissions, and meanwhile, allows any pair of nodes in the network to be connected under an arbitrarily high SINR requirement. We also introduce a hexagon-based scheduling algorithm that allows the CSMA network to be connected. However, it remains a major challenge to find the optimum scheduling algorithm that gives the minimum delay and the maximum capacity under a specific traffic distribution.

APPENDIX I PROOF OF THEOREM 1

A network on a finite area, denoted by $A \subset \Re^2$, can always be obtained from a network on an infinite area $\Re^2$ with the same node density and distribution by removing these nodes outside A. Such removal process will also remove all transmitters outside A. Therefore the interference at a receiver in A is less than or equal to the interference experienced by its counterpart in a network in $\Re^2$. It then suffices to show that the interference in a network in $\Re^2$ is bounded.

Consider that an arbitrary receiver z is located at a Euclidean distance r_0 from its closest transmitter w, which is also the intended transmitter for z. We construct a coordinate system such that the origin of the coordinate system is at w and z is on the $+y$ axis, as shown in Fig. 4.

In a CSMA network, the distance between any two concurrent transmitters is at least R_c. Draw a circle of radius $R_c/2$ centered at each transmitter. Then the two circles centered at two closest transmitters cannot overlap except at a single point. Therefore the problem of determining the maximum interference can be transformed into one that determining the maximum number of equal-radius non-overlapping circles that can be packed into $\Re^2$. The densest circle packing, i.e. fitting the maximum number of non-overlapping circles into $\Re^2$, is obtained by placing the circle centers at the vertices of a hexagonal lattice [31, p. 8], as shown in Fig. 4.
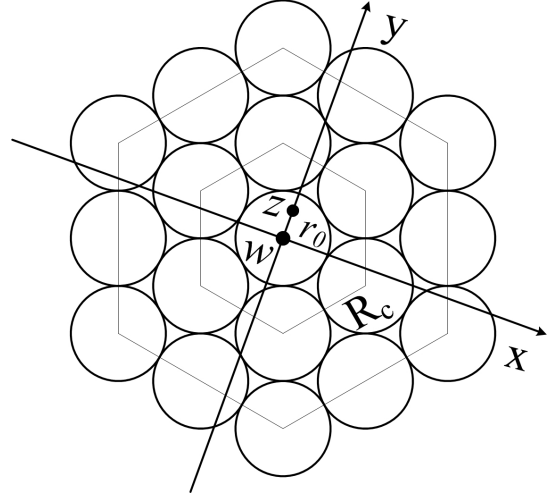

Figure 4. An illustration of the densest equal-circle packing.

Group the vertices of the hexagonal lattice into tiers of increasing distances from the origin. The six vertices of the first tier are within a Euclidean distance R_c to the origin. The $6m$ vertices in the mth tier are located at distances within $((m-1)R_c, mR_c]$ from the origin.

Let I_1 be the interference caused by transmitters, hereinafter referred to as interferers in this section, above the x-axis at node z. Using the triangle inequalities gives $\|x_i - z\| \geq \|x_i\| - r_0$ where x_i is the location of an interferer above the x-axis. Among the $6m$ interferers in the mth group, half of them are located above the x-axis. Among these interferers in the mth group above the x-axis, three of them are at a Euclidean distance of exactly mR_c from the origin and the rest $3(m-1)$ interferers are at Euclidean distances within $[\frac{\sqrt{3}}{2}mR_c, mR_c]$. Hence, we have

$$I_1 \leq \sum_{m=1}^{\infty} \left(\frac{3(m-1)P}{\left(\frac{\sqrt{3}}{2}mR_c - r_0\right)^\alpha} + \frac{3P}{(mR_c - r_0)^\alpha} \right) \quad (19)$$

Look at the first summation in (19). Let $U_m, m = 3, \ldots, \infty$, be random variables uniformly and i.i.d. in $[m - 1/2, m + 1/2]$. It follows from the convexity of $\frac{3(m-1)P}{\left(\frac{\sqrt{3}}{2}mR_c - r_0\right)^\alpha}$ and Jensen's inequality (used in the second step) that

$$\begin{aligned}
&\sum_{m=3}^{\infty} \frac{3(m-1)P}{\left(\frac{\sqrt{3}}{2}mR_c - r_0\right)^\alpha} \\
&= \sum_{m=3}^{\infty} \frac{3(\mathrm{E}(U_m) - 1)P}{\left(\frac{\sqrt{3}}{2}\mathrm{E}(U_m)R_c - r_0\right)^\alpha} \quad (20) \\
&\leq \sum_{m=3}^{\infty} \mathrm{E}\left(\frac{3(U_m - 1)P}{\left(\frac{\sqrt{3}}{2}U_mR_c - r_0\right)^\alpha} \right) \\
&= \sum_{m=3}^{\infty} \int_{m-1/2}^{m+1/2} \frac{3(x-1)P}{\left(\frac{\sqrt{3}}{2}xR_c - r_0\right)^\alpha} dx \\
&= 3P \int_{5/2}^{\infty} (x-1)\left(\frac{\sqrt{3}}{2}xR_c - r_0\right)^{-\alpha} dx
\end{aligned}$$

$$= \frac{4P\left(\frac{5\sqrt{3}}{4}R_c - r_0\right)^{1-\alpha}\left(\frac{\sqrt{3}}{4}(3\alpha-1)R_c - r_0\right)}{R_c^2(\alpha-1)(\alpha-2)} \quad (21)$$

Likewise, we also have $\sum_{m=2}^{\infty} \frac{3P}{(mR_c-r_0)^{\alpha}} \leq \frac{3P\left(\frac{3}{2}R_c-r_0\right)^{1-\alpha}}{(\alpha-1)R_c}$. As a result of the last equation and (19), (21), (4), it follows that $I_1 \leq N_1(r_0)$.

Now we consider the total interference caused by interferers below the x-axis at node z, denoted by I_2. Let $\boldsymbol{x}_i$ be the location of an interferer below the x-axis, it follows from the triangle inequality that $\|\boldsymbol{x}_i - \boldsymbol{z}\| \geq \|\boldsymbol{x}_i\|$. Therefore

$$
\begin{aligned}
I_2 &\leq \sum_{m=1}^{\infty}\left(\frac{3P}{(mR_c)^{\alpha}} + \frac{3(m-1)P}{\left(\frac{\sqrt{3}}{2}mR_c\right)^{\alpha}}\right) \\
&\leq \frac{3P}{R_c^{\alpha}} + \frac{3P(\frac{3}{2})^{1-\alpha}}{(\alpha-1)R_c^{\alpha}} + \frac{3P}{\left(\sqrt{3}R_c\right)^{\alpha}} \\
&\leq +\frac{3P\left(\frac{5}{4}\right)^{1-\alpha}(3\alpha-1)}{(\alpha-1)(\alpha-2)\left(\sqrt{3}R_c\right)^{\alpha}}
\end{aligned} \quad (22)
$$

Combining $I_1 \leq N_1(r_0)$ and (22), Theorem 1 is proved.

APPENDIX II LEMMA 12

Lemma 12 is needed in the proof of Theorem 9. Theorem 11 is used to prove Lemma 12.

Theorem 11. *(Theorem 1 in [32]) Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_j$ be j arbitrary points in $\Re^2$. Let $w_1, w_2, \ldots w_j$ be j positive numbers regarded as weights attached to these points, and define a position vector $\boldsymbol{c}$ by $\sum_{i=1}^{j} w_i \boldsymbol{v}_i = W\boldsymbol{c}$ where $W = \sum_{i=1}^{j} w_i$. Then for an arbitrary point $\boldsymbol{z}$, the following holds: $\sum_{i=1}^{j} w_i \|\boldsymbol{v}_i - \boldsymbol{z}\|^2 = \sum_{i=1}^{j} w_i \|\boldsymbol{v}_i - \boldsymbol{c}\|^2 + W\|\boldsymbol{z} - \boldsymbol{c}\|^2$*

Lemma 12. *Consider a triangular lattice with unit side length and having a vertex located at the origin o. Define the 1st tier of points to be the six points placed at the vertices of the triangular lattice at a distance of 1 to the origin o. Let the mth tier of points be the $6m$ points placed at the vertices of the triangular lattice located at distances within $(m-1, m]$ from the origin o, as shown in Figure 5. The total number of points from the 1st tier to the mth tier then equals to $j = 3m(1+m)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_j$ be the location vectors of these j points and the points are ordered according to their distances to the origin o in a non-decreasing order. For an arbitrary point $\boldsymbol{z}$ located inside the hexagon formed by the 1st tier of six points, the following holds: $\sum_{i=1}^{j} \|\boldsymbol{v}_i - \boldsymbol{z}\|^{-\alpha}$ is minimized when $\boldsymbol{z}$ is located at the origin o.*

Proof: Now we use Theorem 11 to prove Lemma 12. Letting all attached weights w_i equal to 1 and using Theorem 11, for an arbitrary point $\boldsymbol{z}$ located inside the hexagon formed by the 1st tier of six points, we have

$$\sum_{i=1}^{6} \|\boldsymbol{v}_i - \boldsymbol{z}\|^2 = \sum_{i=1}^{6} \|\boldsymbol{v}_i - \boldsymbol{c}\|^2 + 6\|\boldsymbol{z} - \boldsymbol{c}\|^2 \quad (23)$$

where $\boldsymbol{c}$ is given by $\sum_{i=1}^{6} \boldsymbol{v}_i = 6\boldsymbol{c}$. It is clear that $\boldsymbol{c}$ is the centroid of the six points. Since the hexagon has a unit side length, $\|\boldsymbol{v}_i - \boldsymbol{c}\|$ equals to 1. Let $x_i = \|\boldsymbol{v}_i - \boldsymbol{z}\|$ and $y = $
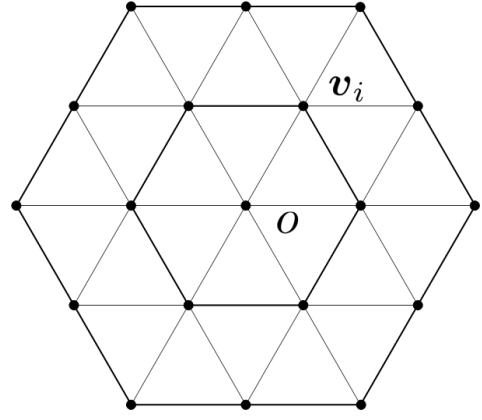

Figure 5. An illustration of a triangular lattice

$\|\boldsymbol{z} - \boldsymbol{c}\|$. The problem in Lemma 12 can then be converted to the following constrained minimization problem:

minimize $\quad f(x_1, \ldots, x_6) = \sum_{i=1}^{6} x_i^{-\alpha}$

subject to $\quad h(x_1, \ldots, x_6) = \sum_{i=1}^{6} x_i^2 - 6 - 6y^2 = 0$

where the constraint is due to (23). Using the method of Lagrange multipliers, we first construct the Lagrangian in the following: $F(x_1, \ldots x_6, \Lambda) = f(x_1, \ldots, x_6) + \Lambda h(x_1, \ldots, x_6)$ where the parameter Λ is known as the Lagrange multiplier. Then find the gradient and set it to zero: $\nabla F(x_1, \ldots x_6, \Lambda) = $
$$\begin{pmatrix} -\alpha x_1^{-\alpha-1} + 2\Lambda x_1 \\ \vdots \\ -\alpha x_6^{-\alpha-1} + 2\Lambda x_6 \\ h(x_1, x_2, \ldots, x_6) \end{pmatrix}^T = \mathbf{0}.$$ Solving the above equation,

it is obtained that $\Lambda = \frac{\alpha}{2}\left(1+y^2\right)^{-\frac{\alpha+2}{2}}$ and $x_1 = x_2 \ldots = x_6 = \left(\frac{2\Lambda}{\alpha}\right)^{\frac{-1}{\alpha+2}} = \left(1+y^2\right)^{\frac{1}{2}}$. Since $x_i = \|\boldsymbol{v}_i - \boldsymbol{z}\|$ denotes the Euclidean distance from $\boldsymbol{v}_i$ to $\boldsymbol{z}$, only when $\boldsymbol{z} = \boldsymbol{c}$, we can have $x_1 = x_2 = \ldots = x_6 = 1$. It follows that the minimum of $f(x_1, x_2, \ldots, x_6)$ is obtained only when $\boldsymbol{z}$ is located at the origin o. Further, for the $6m$ points of the mth tier, using the same method, it can be shown that $\sum_{i=1}^{6m} \|\boldsymbol{v}_i - \boldsymbol{z}\|^{-\alpha}$ is minimized only when $\boldsymbol{z}$ is located at the origin o. The result follows. ∎

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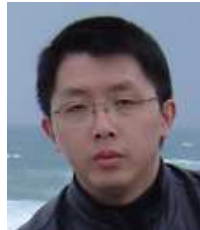
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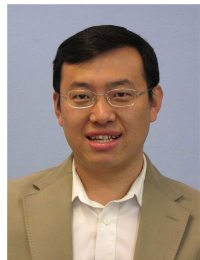
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