Performance Analysis of Distributed Raptor Codes in Wireless Relay Networks

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Abstract—In this paper, we propose a distributed network coding (DNC) scheme based on the Raptor codes for wireless relay networks (WRNs), where a group of source nodes communicate with a single sink through a common relay network in a multi-hop fashion. At the sink, a graph-based Raptor code is formed on the fly. After receiving a sufficient number of encoded packets, the sink begins to decode. The main contributions of this paper are the derivations of upper and lower bit error rate (BER) bounds for the proposed Raptor-based DNC scheme.

Index Terms—Distributed network coding, wireless relay networks, Raptor codes, Quasi-static Rayleigh fading channel, upper and lower bounds.

I. INTRODUCTION

Wireless systems, such as last mile, sensor or community networks, are likely to be a major form of future communications. Next generation wireless communication networks will go beyond point-to-point or point-to-multipoint paradigms of existing cellular networks. They will be based on complex interactions, where individual communication nodes cooperate with one another in order to improve the performance of their own communication and that of the entire network. Cooperative communications [1], [2], based on the use of relay nodes, have emerged as a promising approach to increase spectral and power efficiency, communication coverage and reliability.

As a special channel coding strategy developed for cooperative communication networks, distributed coding techniques attracted significant research interest recently. The distributed code construction concept has been applied to conventional channel coding to form, for example, distributed turbo codes [3], distributed space-time codes [4] and distributed low-density parity-check (LDPC) codes [5]. These developments show that the distributed coding schemes are capable of improving the transmission reliability in cooperative communication networks [1].

Cooperative communication offers a significant advantage in improving communication reliability, increasing spectral

efficiency and reducing energy consumption [1], [2]. A widely used cooperative communication technique is network coding [6]. Network coding is a novel approach to enhance the network capacity [6]. When network coding is employed, the relay nodes are allowed to encode the packets received from multiple source nodes to compress the data, and whenever possible, to reduce the number of transmissions and bandwidth consumption. Prior work shows that network coding achieves the multi-cast network capacity by transmitting linear combinations of received data [7], [8].

Due to the broadcast nature of wireless channels, network coding appears to be a natural fit for cooperative communication networks where multiple source nodes communicate with a common sink through multiple relay nodes. There are a large number of published papers in this area, e.g., [9]–[14]. A distributed network coding scheme for wireless relay networks (WRNs) using Luby-transform (LT) [15] codes, referred to as distributed network coding (DNC) scheme [16], was proposed to reduce the network code design to the problem of designing a Tanner graph code by matching a real-time network topology to a code on graph. LT codes are a class of fountain codes. Fountain codes [17], which are also referred to as rateless codes, can provide reliable transmission by using continuous transmission schemes. Specifically, fountain codes transform a data sequence into a potentially unlimited number of encoded packets, such that it is possible to correctly decode the original data sequence, as long as a number of encoded packets with the length slightly larger than the original data sequence are received.

In the aforementioned works, source nodes are assumed to be able to communicate directly with their common sink. However, in some WRN applications, where direct links between source nodes and the common sink are not always available, communications of data occur in a multi-hop way. In this paper, we propose a DNC scheme for WRNs, where no directly links exist between the source nodes and their common sink, based on Raptor codes [18], in which a conventional outer code is concatenated with an inner LT code to reduce the error floor. The main challenge of introducing Raptor codes into WRNs is to conduct precoding in a decentralized man-

ner [19]. Different decentralized methods have been studied in previous works, such as the random-walk based decentralized fountain codes algorithm given in [20] and the efficient two-hop decentralized precoder described in [21]. In this paper, a WRN with multiple source nodes, multiple relay nodes and a single sink is considered. All the source nodes send their data packets to the relay nodes by using a random access MAC (Media Access Control) protocol. All the relay nodes which participate in the process of transmission form two relay groups, namely, a precoding relay group and an LT-coding relay group. The two relay groups are formed in a decentralized way. The operations of the relay nodes in the LT-coding and precoding relay groups are also decentralized. Each relay node in the LT-coding and precoding relay groups selects packets uniformly at random from its buffer and then performs a linear network coding. At the sink a graph code is formed on the fly.

We derive analytical upper and lower bounds for the bit error rate (BER) of the proposed Raptor-based DNC scheme over Rayleigh fading channels under maximum-likelihood (ML) decoding [22] [23]. These derived bounds are shown to be asymptotically tight with the increasing of the expanding coefficient. As the derived analytical bounds are tight for large SNRs, our ML analysis is of interest and may be used to optimize degree distributions for the proposed scheme [24, Ch. 8, p. 283]. The analytical bounds can also be used to compare the performance of Raptor codes with different parameters.

II. SYSTEM MODEL

Before the introduction of the system model, we first recall the basic knowledge of the Raptor codes.

A. Raptor Codes

Raptor codes are the first class of fountain codes with linear time encoding and decoding. The key idea of Raptor coding is concatenating a traditional error correcting code with an LT code to relax the condition that all input symbols need to be recovered in LT codes. The encoding process is carried out in two phases:

- Encode k input symbols with an (n, k) error correcting block code to form n intermediate symbols.
- Encode the n intermediate symbols with an LT code. Each output symbol is generated by randomly choosing a degree d following the degree distribution $\mu(x)$, and then choosing d distinct input symbols uniformly at random, and operating XOR on them.

Thus, a Raptor code can be specified by parameters $(k, C, \mu(x))$, where C is the (n, k) error correcting block code, called the pre-code, and $\mu(x)$ is the generator polynomial of the degree distribution of the LT code.

Corresponding to the encoding process, the decoding process also can be divided into two phases:

- Recover required fraction of intermediate symbols from the received symbols at the sink.

- Recover all input symbols from the fixed fraction of intermediate symbols, which is recovered at the first phase.

B. Distributed Raptor Codes for Wireless Relay Networks

As shown in Fig. 1, we consider a wireless relay network with K_L source nodes communicating with a single sink through a common relay network. We assume that there are K relay nodes in the relay network. We consider a random access MAC protocol, e.g., Carrier Sense Multiple Access-Collision Avoidance (CSMA-CA). Further, we only consider 3-hop transmissions. We assume that all the relay nodes in the network know previously the degree distributions of the designed distributed Raptor codes.

The relay nodes can be classified into two groups: a precoding relay group $\mathbb{R}_1$, and an LT-coding relay group $\mathbb{R}_2$. Depending on the header information of received packets, a relay node determines which group it belongs to. We assume that we have a priori knowledge of the total number of relay nodes in the precoding relay group and denote this number by K_P. The priori knowledge can be obtained through a priori measurements of the relay network or simply by estimating the number of the relay nodes which are within the transmission range of the source nodes. More detailed description of the classification of the two groups will be given later in this section. Fig. 1 shows the model of the investigated system. Let S_i, $R_{j'}^1$, R_j^2 and D represent the ith source node, the j'th relay node in relay group $\mathbb{R}_1$, the jth relay node in relay group $\mathbb{R}_2$ and the sink, respectively, where $i \in \{1, 2, \ldots, K_L\}$, $j' \in \{1, 2, \ldots, K_P\}$ and $j \in \{1, 2, \ldots, N\}$.
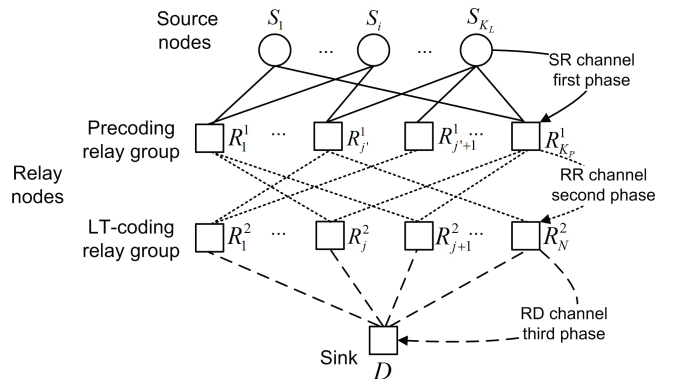

Fig. 1. A network graph used to describe the system model.

We assume that the channels between the source and relay nodes in the precoding relay group, between the relay nodes in the two relay groups, and between the relay nodes in the LT-coding relay group and the sink are quasi-static Rayleigh fading channels and spatially independent. We denote by $\hbar_{i,j'}^{os}$ the fading coefficient of the link between the ith source node S_i and the j'th relay node $R_{j'}^1$ in relay group $\mathbb{R}_1$. Let $\hbar_{j',j}^{to}$ and $\hbar_j^{Dt}$ be the channel fading coefficients of the links between $R_{j'}^1$ and the jth relay node R_j^2 in relay group $\mathbb{R}_2$, and between R_j^2 and the sink D, respectively. The fading coefficients $\hbar_{i,j'}^{os}$,

$\hbar_{j',j}^{to}$ and $\hbar_j^{Dt}$ remain constant over the length of one data packet, but change independently from packet to packet. These fading coefficients have unit mean square values [25], i.e., $E\{|\hbar_{i,j'}^{os}|^2\} = 1$, $E\{|\hbar_{j',j}^{to}|^2\} = 1$ and $E\{|\hbar_j^{Dt}|^2\} = 1$. Here $E\{\cdot\}$ denotes the expectation. We allow the relay nodes to operate with different transmit powers.

For simplicity, we adopt the binary phase shift keying (BPSK) [26] modulation and assume perfect channel state information (CSI) at the receivers. The data delivery from the source nodes to the sink is carried out in four phases, i.e., the broadcast phase, the precoding phase, the LT-coding phase and the data recovery phase.

Broadcast phase: The information sequence of each source node is segmented into a number of equal length data segments at first. Then a header and a cycle redundance check (CRC) code are added to each data segment to form a data packet. Denote by $\mathbf{m}_i \in \mathrm{GF}(2)^{\epsilon \times 1}$, $i \in \{1, 2, \cdots, K_L\}$, the data packet of the ith source node. The header consists of the identity of the node, the number of hops of the transmitted packets, etc. The number of hops for the packets transmitted from source nodes is set to 1.

Precoding phase: In the precoding phase, LDPC codes are used. $\Omega(x)$ is the generator polynomial of the degree distribution of the LDPC codes [18] [27]. The information of the degree distribution $\Omega(x) = \sum_{p=1}^{n} \Omega_p x^p$ is previously stored in the relay nodes. When a relay node receives a packet, it decodes the packet and checks the correctness of the decoded packet by using CRC. If the packet is correct, the relay node then checks the number of hops of the received packet. If the number is equal to 1, the node determines that it belongs to the precoding relay group.

- At first, each relay node in $\mathbb{R}_1$ selects a number p with probability Ω_p according to its pre-stored degree distribution $\Omega(x)$. The received data packet from S_i at $R_{j'}^1$, $(R_{j'}^1 \in \mathbb{R}_1)$, can be written as $\mathbf{y}_{i,j'}^{os} = \hbar_{i,j'}^{os} \mathbf{x}_i + \mathbf{n}_{i,j'}^o$. Here $\mathbf{x}_i = (-1)^{\mathbf{m}_i} \in \{\pm 1\}^{\epsilon}$ is the transmitted data packet from S_i, $\mathbf{n}_{i,j'}^o$ is the additive white Gaussian noise vector of size $\epsilon \times 1$ at $R_{j'}^1$, where ϵ denotes the length of the data packet. Then each relay node in $\mathbb{R}_1$ decodes the received data packets from the source nodes. If the decoded packet is correct, the node then puts the correctly decoded data segments together with its connection information into its buffer.

- After collecting enough number of data segments, i.e., the number of data segments in a relay node's buffer is larger than its selected number p, the relay node selects a number p of data segments from its buffer to perform network coding by using linear combinations in the field of GF(2) [22]. Since $p \ll K_L$, it is with high probability that there are p correctly decoded data segments in the buffer. Thus, we can always assume that this network coding process is achievable. The network-coded data are then augmented with a header and a CRC code to form a network-coded data packet. Denote by $\mathbf{s}_{j'} \in GF(2)^{\epsilon \times 1}$, $j' \in \{1, 2, \ldots, K_P\}$, the network-coded data packet at $R_{j'}^1$. The header contains the connection information

and the number of hops of the packet. The connection information includes the identities of the source nodes which are used to form the network coded packet and the identity of the relay node from which the coded packet is broadcast. The number of hops for the packets transmitted from the relay nodes in precoding group $\mathbb{R}_1$ is set to 2.

- The network-coded data packet is then broadcast at each relay node in $\mathbb{R}_1$ to the network with a probability P_P, where $P_P = \frac{K}{K_P}$. Here $K = \frac{K_L}{r}$ denote the total number of the selected relay nodes in $\mathbb{R}_1$ and r is the desired code rate of the pre-code.

LT-coding phase: In the LT-coding phase, LT codes are used. $\mu(x)$ is the generator polynomial of the degree distribution of the LT codes [15] [18]. The information of the degree distribution $\mu(x) = \sum_q \mu_q x^q$ is previously stored in the relay nodes. If a relay node receives a packet with the number of hops of 2, then it determines that it belongs to the LT-coding relay group. Note that the relay node belonging to the LT-coding relay group may also belong to the precoding relay group. But in different phases, it performs its operation according to different degree distributions.

- Each relay node in $\mathbb{R}_2$ first randomly selects a number of q with probability μ_q according to the pre-stored degree distribution $\mu(x)$. The received data packet from $R_{j'}^1$ at R_j^2 can be represented as $\mathbf{y}_{j',j}^{to} = \hbar_{j',j}^{to} \mathbf{x}_{j'}^o + \mathbf{n}_{j',j}^t$. Here $\mathbf{x}_{j'}^o = (-1)^{\mathbf{s}_{j'}} \in \{\pm 1\}^{\epsilon}$ is the modulated data packet transmitted from $R_{j'}^1$. $\mathbf{n}_{j',j}^t$ is the additive white Gaussian noise vector of size $\epsilon \times 1$ at R_j^2. Then each relay node in $\mathbb{R}_2$ decodes the received data packets from the relay nodes in $\mathbb{R}_1$, checks the correctness using CRC code, and puts the correctly decoded data segments together with their connection information into its buffer.

- After collecting enough number of data segments, i.e., the number of data segments in the relay node's buffer is larger than its previously selected number q, each relay node in $\mathbb{R}_2$ randomly selects a number q of data segments from its buffer to perform network coding by using linear combinations in the field of GF(2). Since $q \ll K$, we can always assume that this network coding process is possible to achieve. The formed network coded data segment is then appended with a CRC code and a header. The header contains the connection information and the number of hops of the packet. The connection information includes the identities of the source nodes and the relay nodes in $\mathbb{R}_1$ which are used to form the network coded data segment, it also includes the identity of the relay node in $\mathbb{R}_2$ from which the packet is transmitted. The number of hops for the packets transmitted from any nodes in $\mathbb{R}_2$ is set to 3.

- The network-coded packet is then broadcast to the sink.

Data recovery phase: The sink knows whether the packets are from LT coding group by checking the number of hops of the received packets, i.e., if the number of hops of a received packet equals to 3, then it is from the LT-coding group. The received data packet from the jth relay node in $\mathbb{R}_2$ at the sink

D can be expressed as $\mathbf{y}_j^{Dt} = \hbar_j^{Dt}\mathbf{x}_j^t + \mathbf{n}_j^D$, where $\mathbf{x}_j^t \in \{\pm 1\}^\epsilon$ is the modulated data packet transmitted from R_j^2, and $\mathbf{n}_j^D$ is the additive white Gaussian noise vector of size $\epsilon \times 1$. After receiving Q data packets from the relay nodes in relay group $\mathbb{R}_2$, where $Q \geq K$, the sink begins to decode by using the ML algorithm, based on the soft information of the received data packets.

Since a small header, which contains the connection information, is added in each packet when each relay node forwards data, the sink knows how the checks are formed and can accordingly replicate the code on graph, as shown in Fig. 2. The formed code graph consists of K_L input data packets, corresponding to the K_L source nodes, K precoding data packets, corresponding to the K relay nodes which are selected to transmit network-coded packets in $\mathbb{R}_1$, and Q output data packets. The length of each header is much shorter than that of the data packet, therefore, the throughput loss due to the header can be neglected.
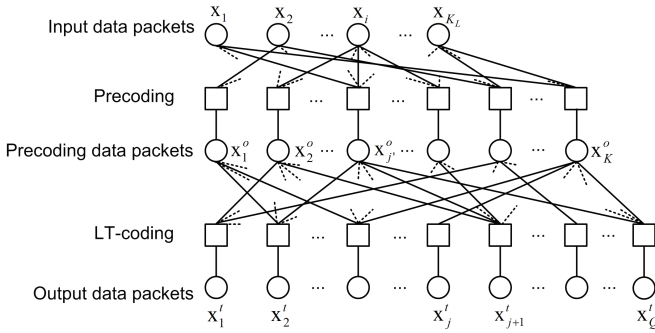

Fig. 2. A Tanner graph of Raptor code derived from the system model.

If the data packets cannot be recovered completely at the sink, the sink simply sends a data request to the relay nodes in relay group $\mathbb{R}_2$ to ask for another network-coded data packet. After receiving the request from the sink, each relay node in the LT-coding relay group transmits a new formed data packet to the sink with the probability of $\frac{1}{N}$, where N is the number of relay nodes in the LT-coding relay group. We assume that this number can be estimated. As a result, every time the sink sends a transmit request to the relay nodes, only one relay node responds to the request. The responding relay node is regarded as the $(Q+1)$th node. This process runs until all the information can be recovered at the sink. Since the relay nodes are asked for data transmission one by one, redundant transmissions are avoided.

III. Upper and Lower Bounds on the Bit Error Probability of Raptor Codes

In this section, we derive both upper and lower bounds on the bit error probability of the proposed finite-length Raptor-based DNC scheme under ML decoding over Rayleigh fading channels.

The data packets from all the source nodes are arranged into a matrix $\mathbf{\Lambda} = [\mathbf{m}_1, \mathbf{m}_2, \cdots, \mathbf{m}_{K_L}] \in \mathrm{GF}(2)^{\epsilon \times K_L}$. All the data packets formed at all the relay nodes in $\mathbb{R}_1$ during the precoding phase are grouped into a matrix $\mathbf{\Lambda}_s = [\mathbf{s}_1, \mathbf{s}_2, \cdots, \mathbf{s}_K] \in \mathrm{GF}(2)^{\epsilon \times K}$, where $\mathbf{s}_{j'} \in \mathrm{GF}(2)^{\epsilon \times 1}$, $j' \in \{1, 2, \cdots, K\}$, is the network-coded data packet of the j'th selected relay node in $\mathbb{R}_1$. The output data packets of the relay nodes in $\mathbb{R}_1$ and $\mathbb{R}_2$ are formed as described in Section II, according to the pre-determined degree distributions $\Omega(x)$ and $\mu(x)$, respectively.

The network-coded data packets formed at $\mathbb{R}_2$ in the LT-coding phase are then transmitted over quasi-static Rayleigh fading channels to the sink D. Let Q denote the number of data packets collected at the sink. Then Q can be expressed as $Q = \eta K = \frac{\eta}{r}K_L$, where the expanding coefficient η is a real number equal to or greater than one.

The output from the Q relay nodes in $\mathbb{R}_2$ can be represented by $\mathbf{V} = \mathbf{\Lambda}_s\mathbf{G} = \mathbf{\Lambda}\mathbf{G}_L\mathbf{G}$, where $\mathbf{V} \in \mathrm{GF}(2)^{\epsilon \times Q}$. $\mathbf{G}_L$ is a randomly generated $K_L \times K$ binary matrix formed at $\mathbb{R}_1$, which is associated with the graph code used in the precoding phase of the Raptor code. $\mathbf{G}$ is a randomly generated $K \times Q$ binary matrix formed at $\mathbb{R}_2$, which is associated with the LT code. The jth column of $\mathbf{V}$ is the transmitted data packet from the jth relay node in $\mathbb{R}_2$. Since the binary information sequences of the source nodes are assumed to be i.i.d, we only consider an arbitrary row of $\mathbf{\Lambda}$ in the analysis and denoted by $\mathbf{m}$. Let $\mathbf{v}$ be the corresponding row of $\mathbf{V}$ when $\mathbf{m}$ is chosen as the input, i.e., $\mathbf{v} = \mathbf{m}\mathbf{G}_L\mathbf{G}$.

The (i, j')th element of $\mathbf{G}_L(i, j')$ is equal to one if the ith source node's data is used to generate the output data at the j'th selected relay node in $\mathbb{R}_1$, otherwise it is zero. In a similar way, $\mathbf{G}(j', j) = 1$, if the data of the j'th selected relay node in $\mathbb{R}_1$ is used to generate the output data of the jth relay node in $\mathbb{R}_2$, otherwise it is zero. Since the both applied codes are linear, their performance is independent of the transmitted information sequence. Thus, we can assume that the all zero sequence is transmitted, i.e., $\mathbf{m} = \mathbf{0}$. The error information sequence is denoted by $\mathbf{e}$.

A. Upper Bound on the Bit Error Probability of Raptor Codes

The probability of decoding error for the Raptor codes is calculated by the probability that the information cannot be recovered correctly either by the LT decoder or by the precoding decoder. We assume that after the LT decoder, the error sequence is $\mathbf{e}$. The probability that the precoding decoder fails to recover the information can be computed as

$$P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}) = \sum_{t=1}^K P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t), \quad (1)$$

where $\mathbf{H}$ is the parity check matrix of the generator matrix $\mathbf{G}_L$. $\mathbf{H}^T$ is the transpose matrix of the parity check matrix $\mathbf{H}$.

The term in the right side of (1) can be expressed as

$$P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t)$$
$$= \left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})} P_r((\mathbf{e} \otimes \mathbf{h}) = 0, w(\mathbf{e}) = t | w(\mathbf{h})) \right)^{K - K_L} \quad (2)$$

where $\mathbf{h}$ is a column vector of the matrix $\mathbf{H}^T$, $\Omega_{w(\mathbf{h})}$ is the probability that $\mathbf{h}$ has weight $w(\mathbf{h})$.

When $t = 1$, the probability $P_r(\mathbf{e} \otimes \mathbf{h} = 0 | w(\mathbf{h}), w(\mathbf{e}) = t))$ in (2) is equal to the probability that all the elements in a special row of $\mathbf{H}^T$ are zeros, and the special row corresponds to the position where the nonzero value occurs in the error sequence $\mathbf{e}$. When $t > 1$, let $\zeta = \{i_1, i_2, \cdots, i_t\}$ be the set of size $|\zeta| = w(\mathbf{e}) = t \leq K$, whose elements are the positions where all the nonzero values exist in $\mathbf{e}$. Let $\mathbf{h}(\zeta)$ be the corresponding subvector of $\mathbf{h}$. Then $P_r(\mathbf{e} \otimes \mathbf{h} = 0 | w(\mathbf{h}), w(\mathbf{e}) = t))$ is equal to the probability that $\mathbf{h}(\zeta)$ contains an even number of ones. Therefore

$$P_r(\mathbf{e} \otimes \mathbf{h} = 0, w(\mathbf{e}) = t | w(\mathbf{h}))$$
$$= \frac{\sum_{\alpha = even, 0 \leq \alpha \leq t} \binom{t}{\alpha} \binom{K-t}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}} = \varsigma_t^{R, w(\mathbf{h})}. \quad (3)$$

Note that (1) depends on $\mathbf{e}$ only through $w(\mathbf{e}) = t$. Also, there are $2^K - 1$ nonzero error sequences $\mathbf{e}$, of which $\binom{K}{t}$ have weight t. Taking these factors into account, we have that the information bit error probability of the Raptor code can be upper bounded by $P_b < \min\{1, \xi_U^{Raptor}\}$, where

$$\xi_U^{Raptor} = \sum_{t=1}^{K} \frac{t}{K} \binom{K}{t} \Big[\xi_U^t (1 - \xi_U)^{K-t}$$
$$\cdot P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t) \Big]. \quad (4)$$

The scaling factor t/K in (4) is the number of erroneous bits per information sequence in a weight-t information error sequence. Therefore, we can get the BER upper bound of Raptor codes which can be described as:

Theorem 1: Consider a Raptor code with parameters $\Omega(x)$, $\mu(x)$, K_L, K, Q, r and expanding coefficient η. Its upper BER bound under ML decoding over a fast Rayleigh fading channel is $P_b < \min\{1, \xi_U^{Raptor}\}$, where ξ_U^{Raptor} is given by

$$\xi_U^{Raptor} = \sum_{t=1}^{K} \frac{t}{K} \binom{K}{t} \xi_U^t (1 - \xi_U)^{K-t}$$
$$\cdot \left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})} \varsigma_t^{R, w(\mathbf{h})} \right)^{K-K_L}$$

where

$$\varsigma_t^{R, w(\mathbf{h})} = \frac{\sum_{\alpha = even, 0 \leq \alpha \leq t} \binom{t}{\alpha} \binom{K-t}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}},$$

where $\mathbf{h}$ is a column vector of the matrix $\mathbf{H}^T$, $\mathbf{H}^T$ is the transpose matrix of the parity check matrix $\mathbf{H}$, and $\Omega_{w(\mathbf{h})}$ is the probability that $\mathbf{h}$ has weight $w(\mathbf{h})$.

At the sink, the received data packets form a data matrix $\mathbf{V}' = [\mathbf{y}_1^{Dt}, \ldots, \mathbf{y}_j^{Dt}, \ldots, \mathbf{y}_Q^{Dt}]$. The decoding process at the sink is in a line-by-line order. And due to the fact that the fading coefficients remain constant over the length of one data packet but change independently among different packets, the decoding process at the sink can be seen as over a fast Rayleigh fading channel but with perfect CSI. ξ_U is the upper

BER bound of the LT codes over the special fast Rayleigh fading channel under ML decoding [16]. It can be written as

$$\xi_U < \min \Bigg\{ 1, \sum_{\kappa=1}^{K} \frac{\kappa}{K} \binom{K}{\kappa} \Bigg[\left(\sum_{w(\mathbf{c})} \mu_{w(\mathbf{c})} \varsigma_\kappa^{w(\mathbf{c})} \right)^Q$$
$$+ \sum_{k=1}^{Q} \frac{1}{\pi} \binom{Q}{k} \beta_\kappa^k (1 - \beta_\kappa)^{Q-k}$$
$$\cdot \int_0^{\pi/2} \left(\frac{1}{1 + \frac{E_s^k}{N_0 \sin^2(\theta)}} \right)^k d\theta \Bigg] \Bigg\},$$

where

$$\varsigma_\kappa^{w(\mathbf{c})} = \frac{\sum_{\alpha = even, \alpha \leq \kappa} \binom{\kappa}{\alpha} \binom{K-\kappa}{w(\mathbf{c})-\alpha}}{\binom{K}{w(\mathbf{c})}},$$
$$\beta_\kappa = \sum_{w(\mathbf{c})} \mu_{w(\mathbf{c})} (1 - \varsigma_\kappa^{w(\mathbf{c})}).$$

B. Lower Bound on the Bit Error Probability of Raptor Codes

The error probability of Raptor codes can be lower bounded by following the Bonferroni inequality [28]. The probability can be computed as

$$P_b = \sum_{t=1}^{K} \frac{t}{K} \binom{K}{t} \xi_L^t (1 - \xi_L)^{K-t} P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t)$$
$$- \frac{1}{2} \sum_{t=1}^{K} \frac{t}{K} \binom{K}{t} \xi_U^t (1 - \xi_U)^{K-t}$$
$$\cdot P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, \mathbf{e}'\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t), \quad (5)$$

where $\mathbf{e}'$ is another nonzero error sequence different from the error sequence $\mathbf{e}$.

Let $z \in GF(2)^K$ and $I_z = \{i_{z_1}, i_{z_2}, \ldots, i_{z_\tau}\}$ be the set of indices such that $z(t) = 1$ for $t \in I_z$, otherwise $z(t) = 0$. Define three binary vectors z_0, z_1 and z_2 as $z_0(t) = 1$ if and only if $t \in I_{z_0} := I_e \cap I_{e'}$, $z_1(t) = 1$ if and only if $t \in I_{z_1} := I_e \backslash I_{e'}$ and $z_2(t) = 1$ if and only if $t \in I_{z_2} := I_{e'} \backslash I_e$. Let τ_0, τ_1 and τ_2 be the size of the sets I_{z_0}, I_{z_1} and I_{z_2}, respectively. Then, $P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, \mathbf{e}'\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t)$ in (5) can be expressed as

$$P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, \mathbf{e}'\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t)$$
$$= \sum_{\tau_0=0}^{t} \sum_{\tau_1=t-\tau_0}^{K-\tau_0} \sum_{\tau_2=0}^{K-\tau_0} \frac{\binom{t}{\tau_0}}{2^t} \frac{\binom{K-\tau_0}{t-\tau_0}}{2^{K-\tau_0}} \frac{\binom{K-\tau_0}{\tau_2}}{2^{K-\tau_0}} \Bigg(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})}$$
$$\cdot P_r(\mathbf{e} \otimes \mathbf{h} = \mathbf{0}, \mathbf{e}' \otimes \mathbf{h} = \mathbf{0}, w(\mathbf{e}) = t | \tau_0, \tau_1, \tau_2) \Bigg)^{K-K_L} \quad (6)$$

Corresponding to the three sets I_{z_0}, I_{z_1} and I_{z_2}, each column of the matrix $\mathbf{H}^T$, $\mathbf{h}$, can be divided into three parts, $\mathbf{h}_{\tau_0}$, $\mathbf{h}_{\tau_1}$ and $\mathbf{h}_{\tau_2}$. Let $\mathbf{h}_{\tau_0}$ be the subvector of $\mathbf{h}$ such that all the elements of this subvector are selected from $\mathbf{h}$ according to the indices in set I_{z_0}. The length of $\mathbf{h}_{\tau_0}$ is τ_0. The same operation is applied to the formation of $\mathbf{h}_{\tau_1}$ and $\mathbf{h}_{\tau_2}$, in which the elements are selected according to the indices in set I_{z_1}

and I_{z_2}, and have length τ_1 and τ_2, respectively. Therefore, the conditional probability of the last product term in (10) can be computed as

$$
\begin{aligned}
&P_r(\mathbf{e} \otimes \mathbf{h} = \mathbf{0}, \mathbf{e}' \otimes \mathbf{h} = \mathbf{0}, w(\mathbf{e}) = t | \tau_0, \tau_1, \tau_2) \\
&= P_r(\mathbf{e}_{\tau_0} \otimes \mathbf{h}_{\tau_0} = \mathbf{0}) P_r(\mathbf{e}_{\tau_1} \otimes \mathbf{h}_{\tau_1} = \mathbf{0}) \\
&\quad \cdot P_r(\mathbf{e}'_{\tau_2} \otimes \mathbf{h}_{\tau_2} = \mathbf{0}) + P_r(\mathbf{e}_{\tau_0} \otimes \mathbf{h}_{\tau_0} = \mathbf{1}) \\
&\quad P_r(\mathbf{e}_{\tau_1} \otimes \mathbf{h}_{\tau_1} = \mathbf{1}) P_r(\mathbf{e}'_{\tau_2} \otimes \mathbf{h}_{\tau_2} = \mathbf{1}) \\
&= \sum_{w(\mathbf{h}_{\tau_0})=0}^{\min\{\tau_0, w(\mathbf{h})\}} \sum_{w(\mathbf{h}_{\tau_1})=0}^{\min\{\tau_1, w(\mathbf{h})-w(\mathbf{h}_{\tau_0})\}} \\
&\quad \sum_{w(\mathbf{h}_{\tau_2})=\min\{\tau_2, w(\mathbf{h})-w(\mathbf{h}_{\tau_0})-w(\mathbf{h}_{\tau_1})\}} \\
&\quad \cdot \frac{\binom{\tau_0}{w(\mathbf{h}_{\tau_0})}}{2^{\tau_0}} \frac{\binom{\tau_1}{w(\mathbf{h}_{\tau_1})}}{2^{\tau_1}} \frac{\binom{\tau_2}{w(\mathbf{h}_{\tau_2})}}{2^{\tau_2}} \\
&\quad \cdot \Big[A(w(\mathbf{h}_{\tau_0}), \tau_0) A(w(\mathbf{h}_{\tau_1}), \tau_1) A(w(\mathbf{h}_{\tau_2}), \tau_2) \\
&\quad + \overline{A}(w(\mathbf{h}_{\tau_0}), \tau_0) \overline{A}(w(\mathbf{h}_{\tau_1}), \tau_1) \overline{A}(w(\mathbf{h}_{\tau_2}), \tau_2) \Big], \quad (7)
\end{aligned}
$$

where $\mathbf{e}_{\tau_0}$ and $\mathbf{e}_{\tau_1}$ are subvectors of $\mathbf{e}$ with length τ_0 and τ_1, respectively. The subvector $\mathbf{e}_{\tau_0}$ is formed by selecting elements from $\mathbf{e}$ according to the indices in set I_{z_0}, and the elements in $\mathbf{e}_{\tau_1}$ are selected by the same way according to the indices in set I_{z_1}. All the elements in $\mathbf{e}_{\tau_0}$ and $\mathbf{e}_{\tau_1}$ are in one-to-one correspondence with that of $\mathbf{h}_{\tau_0}$ and $\mathbf{h}_{\tau_1}$, respectively. $\mathbf{e}'_{\tau_2}$ is the subvector of $\mathbf{e}'$ with length τ_2, and the elements of $\mathbf{e}'_{\tau_2}$ are selected from $\mathbf{e}'$ according to the indices in set I_{z_2}. The same as the case of $\mathbf{e}_{\tau_0}$ and $\mathbf{e}_{\tau_1}$, all the elements in $\mathbf{e}'_{\tau_2}$ are in one-to-one correspondence with that of $\mathbf{h}_{\tau_2}$. $w(\mathbf{h}_{\tau_p})$ is the Hamming weight of $\mathbf{h}_{\tau_p}$. Denote by $A(w(\mathbf{h}_{\tau_p}), \tau_p)$ the probability that $z_p \otimes \mathbf{h}_{\tau_p} = 0$, and $\overline{A}(w(\mathbf{h}_{\tau_p}), \tau_p)$ the probability that $z_p \otimes \mathbf{h}_{\tau_p} = 1$. $A(w(\mathbf{h}_{\tau_p}), \tau_p)$ and $\overline{A}(w(\mathbf{h}_{\tau_p}), \tau_p)$ can be computed respectively as

$$
\begin{aligned}
&A(w(\mathbf{h}_{\tau_p}), \tau_p) \\
&= P_r(z_p \otimes \mathbf{h}_{\tau_p} = 0 | w(\mathbf{h}), w(\mathbf{h}_{\tau_2}), \tau_p) \\
&= \frac{\sum_{\alpha=even, 0 \le \alpha \le \min\{\tau_p, w(\mathbf{h}_{\tau_p})\}} \binom{\tau_p}{\alpha} \binom{K-\tau_p}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}}, \\
&\overline{A}(w(\mathbf{h}_{\tau_p}), \tau_p) \\
&= P_r(z_p \otimes \mathbf{h}_{\tau_p} = 1 | w(\mathbf{h}), w(\mathbf{h}_{\tau_2}), \tau_p) \\
&= \frac{\sum_{\alpha=odd, \alpha \le \min\{\tau_p, w(\mathbf{h}_{\tau_p})\}} \binom{\tau_p}{\alpha} \binom{K-\tau_p}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}}. \quad (8)
\end{aligned}
$$

We note that (5) depends on $\mathbf{e}$ only through $w(\mathbf{e}) = t$. Also, there are $2^K - 1$ nonzero error sequences $\mathbf{e}$, of which $\binom{K}{t}$ have weight t. Taking these factors into account, we have that the bit error probability of the Raptor code can be lower bounded by $P_b > \max\{0, \xi_L^{Raptor}\}$, where

$$
\xi_L^{Raptor} = \sum_{t=1}^K \frac{t}{K} \binom{K}{t} \xi_L^t (1-\xi_L)^{K-t}
$$

$$
\begin{aligned}
&\cdot P_r(w(\mathbf{e}\mathbf{H}^T) = 0, w(\mathbf{e}) = t) \\
&- \frac{1}{2} \sum_{t=1}^K \frac{t}{K} \binom{K}{t} \xi_U^t (1-\xi_U)^{K-t} \\
&\cdot P_r(\mathbf{e}\mathbf{H}^T = \mathbf{0}, \mathbf{e}'\mathbf{H}^T = \mathbf{0}, w(\mathbf{e}) = t). \quad (9)
\end{aligned}
$$

The scaling factor t/K in (9) is the number of erroneous information bits per information symbol in a weight-t information error sequence. Therefore, we can get the BER lower bound on the bit error probability of Raptor codes which can be described as

Theorem 2: Consider a Raptor code with parameters $\Omega(x)$, $\mu(x)$, K_L, K, Q, r and expanding coefficient η. Its lower BER bound under ML decoding over a fast Rayleigh fading channel is $P_b > \max\{0, \xi_L^{Raptor}\}$, where ξ_L^{Raptor} is given by

$$
\begin{aligned}
&\xi_L^{Raptor} \\
&= \sum_{t=1}^K \frac{t}{K} \binom{K}{t} \xi_L^t (1-\xi_L)^{K-t} \left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})} \varsigma_t^{R, w(\mathbf{h})} \right)^{K-K_L} \\
&- \frac{1}{2} \sum_{t=1}^K \frac{t}{K} \binom{K}{t} \xi_U^t (1-\xi_U)^{K-t} \sum_{\tau_0=0}^t \sum_{\tau_1=t-\tau_0} \sum_{\tau_2=0}^{K-\tau_0} \\
&\quad \cdot \frac{\binom{t}{\tau_0}}{2^t} \frac{\binom{K-\tau_0}{\tau_1}}{2^{K-\tau_0}} \frac{\binom{K-\tau_0}{\tau_2}}{2^{K-\tau_0}} \left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})} \zeta(\tau_0, \tau_1, \tau_2, t) \right)^{K-K_L},
\end{aligned}
$$

where ξ_L is the lower BER bound of the LT codes over the special fast Rayleigh fading channel under ML decoding. It is given by the probability that only one bit in the decoded sequence is not recovered correctly [18].

$$
\begin{aligned}
&\varsigma_t^{R, w(\mathbf{h})} \\
&= \frac{\sum_{\alpha=even, 0 \le \alpha \le t} \binom{t}{\alpha} \binom{K-t}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}}, \\
&\zeta(\tau_0, \tau_1, \tau_2, t) \\
&= \sum_{w(\mathbf{h}_{\tau_0})=0}^{\min\{\tau_0, w(\mathbf{h})\}} \sum_{w(\mathbf{h}_{\tau_1})=0}^{\min\{\tau_1, w(\mathbf{h})-w(\mathbf{h}_{\tau_0})\}} \\
&\quad \sum_{w(\mathbf{h}_{\tau_2})=\min\{\tau_2, w(\mathbf{h})-w(\mathbf{h}_{\tau_0})-w(\mathbf{h}_{\tau_1})\}} \\
&\quad \cdot \frac{\binom{\tau_0}{w(\mathbf{h}_{\tau_0})}}{2^{\tau_0}} \frac{\binom{\tau_1}{w(\mathbf{h}_{\tau_1})}}{2^{\tau_1}} \frac{\binom{\tau_2}{w(\mathbf{h}_{\tau_2})}}{2^{\tau_2}} \\
&\quad \cdot \Big[A(w(\mathbf{h}_{\tau_0}), \tau_0) A(w(\mathbf{h}_{\tau_1}), \tau_1) A(w(\mathbf{h}_{\tau_2}), \tau_2) \\
&\quad + \overline{A}(w(\mathbf{h}_{\tau_0}), \tau_0) \overline{A}(w(\mathbf{h}_{\tau_1}), \tau_1) \overline{A}(w(\mathbf{h}_{\tau_2}), \tau_2) \Big], \\
&A(w(\mathbf{h}_{\tau_p}), \tau_p) \\
&= \frac{\sum_{\alpha=even, 0 \le \alpha \le \min\{\tau_p, w(\mathbf{h}_{\tau_p})\}} \binom{\tau_p}{\alpha} \binom{K-\tau_p}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}}, \\
&\overline{A}(w(\mathbf{h}_{\tau_p}), \tau_p) \\
&= \frac{\sum_{\alpha=odd, \alpha \le \min\{\tau_p, w(\mathbf{h}_{\tau_p})\}} \binom{\tau_p}{\alpha} \binom{K-\tau_p}{w(\mathbf{h})-\alpha}}{\binom{K}{w(\mathbf{h})}},
\end{aligned}
$$

and $\mathbf{h}_{\tau_p}$ is the subvector of $\mathbf{h}$, τ_p is the length of $\mathbf{h}_{\tau_p}$, and $p \in \{0, 1, 2\}$.

IV. NUMERICAL AND SIMULATION RESULTS

In this section, we present numerical and simulation results for the LT-based DNC scheme and the proposed Raptor-based DNC scheme over quasi-static Rayleigh fading channels. Due to the time-varying channels of the wireless network, within each transmission round, a Raptor code is generated on-the-fly to match the instantaneous network topology. Thus, the BER performance of an ensemble of codes is analyzed. We consider the case of $K = 100$, $K_L = 50$ and 98, which correspond to the pre-code rate $r = 0.5$ and 0.98, respectively. We evaluate the upper and lower bounds on the bit error probability under ML decoding by using the degree distribution $\Omega(x) = rx + (1 - r)x^4$ for the pre-code in the precoding phase and the degree distribution described in [18] for the LT code in the LT-coding phase:

$$
\begin{aligned}
\mu(x) &= 0.007969x + 0.493570x^2 + 0.166220x^3 \\
&\quad + 0.072646x^4 + 0.082558x^5 + 0.056058x^8 \\
&\quad + 0.037229x^9 + 0.055590x^{19} + 0.025023x^{65} \\
&\quad + 0.003135x^{66}.
\end{aligned} \tag{10}
$$

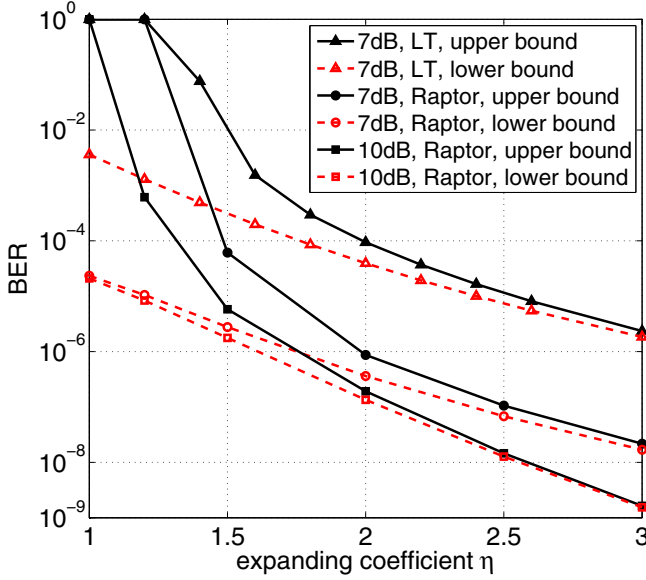
Fig. 3. BER bounds of constructed LT and Raptor codes ensemble over the Rayleigh fading channel with $K_L = 98$, $K = 100$, $E_s/N_0 = 7$dB and 10dB, respectively.

Fig. 3 shows the BER bounds of the constructed LT and Raptor codes ensemble at an average E_s/N_0 of 7dB and 10 dB, respectively. The information length of the LDPC codes is $K_L = 98$, and that of the LT codes is $K = 100$, which correspond to 98 source nodes and 100 relay nodes in $\mathbb{R}_1$, respectively. It can be seen from the figure that as E_s/N_0 increases, e.g., E_s/N_0 changes from 7dB to 10dB, the upper and lower BER bounds of the proposed DNC scheme

based on the Raptor codes become tighter, especially when the expanding coefficient is large enough, e.g., $\eta \geq 1.5$. This phenomenon can be explained by examining the difference between the upper and lower bounds given by *Theorem 1* and *Theorem 2* in Section III, respectively. The difference between the two bounds can be represented by

$$
\begin{aligned}
&\xi_U^{Raptor} - \xi_L^{Raptor} \\
&= \sum_{w(\mathbf{e})=t=1}^{K} \frac{t}{K}\binom{K}{t}\Bigg[\left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})}\varsigma_t^{R,w(\mathbf{h})} \right)^{K-K_L} \\
&\quad \cdot \left(\xi_U^t(1-\xi_U)^{K-t} - \xi_L^t(1-\xi_L)^{K-t} \right) \\
&\quad - \frac{1}{2}\xi_U^t(1-\xi_U)^{K-t} \\
&\quad \cdot \sum_{\tau_0=0}^{t}\sum_{\tau_1=t-\tau_0}\sum_{\tau_2=0}^{K-\tau_0} \frac{\binom{t}{\tau_0}}{2^t}\frac{\binom{K-\tau_0}{\tau_1}}{2^{K-\tau_0}}\frac{\binom{K-\tau_0}{\tau_2}}{2^{K-\tau_0}} \\
&\quad \cdot \left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})}\zeta(\tau_0,\tau_1,\tau_2,t) \right)^{K-K_L}\Bigg]. \quad (11)
\end{aligned}
$$

We only consider the effect from E_s/N_0 to $\xi_U^{Raptor} - \xi_L^{Raptor}$. Let $B(t)$ and $A(t)$ denote the terms of $\sum_{\tau_0=0}^{t}\sum_{\tau_1=t-\tau_0}\sum_{\tau_2=0}^{K-\tau_0} \frac{\binom{t}{\tau_0}}{2^t}\frac{\binom{K-\tau_0}{\tau_1}}{2^{K-\tau_0}}\frac{\binom{K-\tau_0}{\tau_2}}{2^{K-\tau_0}} \left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})}\zeta(\tau_0,\tau_1,\tau_2,t) \right)^{K-K_L}$ and $\left(\sum_{w(\mathbf{h})} \Omega_{w(\mathbf{h})}\varsigma_t^{R,w(\mathbf{h})} \right)^{K-K_L}$ in (11), respectively. These two terms do not depend on E_s/N_0. Therefore, (11) can be rewritten as

$$
\begin{aligned}
&\xi_U^{Raptor} - \xi_L^{Raptor} \\
&= \sum_{w(\mathbf{e})=t=1}^{K} \frac{t}{K}\binom{K}{t}\Bigg[\left(A(t) + \frac{1}{2}B(t) \right)\xi_U^t(1-\xi_U)^{K-t} \\
&\quad - A(t)\xi_L^t(1-\xi_L)^{K-t}\Bigg].
\end{aligned} \tag{12}
$$

Let $f(\xi_\tau)$ denote $\xi_\tau^t(1-\xi_\tau)^{K-t}$ in (12), where $\tau \in \{U, L\}$. From Fig. 3, it can be seen that when the expanding coefficient $\eta \geq 1.5$, the values of both the upper and lower BER bounds of LT codes are smaller than 10^{-2}. Fig. 4 illustrates the change of $f(\xi_\tau)$ with ξ_τ when $\xi_\tau \leq 0.01$. In this figure, green and blue lines denote the cases when $t = 1$ and 100, respectively. Red lines between the green and blue ones denote the cases when $1 < t < 100$.

From Fig. 4 we can see that when $\xi_\tau \leq 0.01$, $f(\xi_\tau)$ is a monotonically increasing function. Since ξ_U is always bigger than ξ_L when $\eta \leq 1.5$, $f(\xi_U)$ can be represented by the function $f(\xi_L)$, that is $f(\xi_U) = f(\xi_L) + \Delta$, where Δ represents the difference between $f(\xi_U)$ and $f(\xi_L)$. Thus, (12) becomes

$$
\begin{aligned}
&\xi_U^{Raptor} - \xi_L^{Raptor} \\
&= \sum_{w(\mathbf{e})=t=1}^{K} \frac{t}{K}\binom{K}{t}\left(\frac{1}{2}B(t)f(\xi_U) + A(t)\Delta \right). \quad (13)
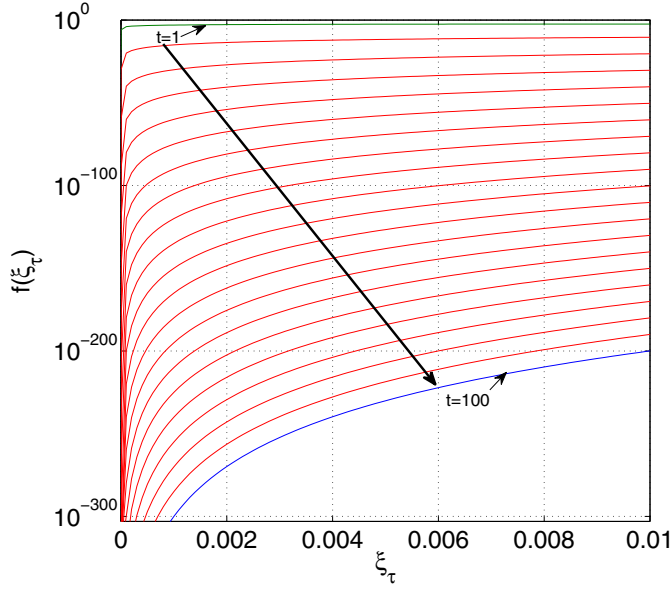\end{aligned}
$$

Fig. 4. $f(\xi_\tau)$ with $K = 100$.

As the E_s/N_0 increases, ξ_U described in *Theorem 1* becomes smaller. Since $f(\xi_U) = \xi_U^t(1-\xi_U)^{K-t}$ is a monotonically increasing function in the region of $0 \leq \xi_U \leq 0.01$, $f(\xi_U)$ becomes smaller with the increases of E_s/N_0.

From the *Theorem 1* and the *Lemma*, we can express ξ_U as $\xi_U = \xi_L + \Delta_o$, where

$$\Delta_o = \sum_{\kappa=2}^{K} \frac{\kappa}{K}\binom{K}{\kappa}\left[\left(\sum_{w(\mathbf{c})}\mu_{w(\mathbf{c})}\varsigma_\kappa^{w(\mathbf{c})}\right)^Q \right.$$
$$+ \sum_{k=1}^{Q}\frac{1}{\pi}\binom{Q}{k}\beta_\kappa^k(1-\beta_\kappa)^{Q-k}$$
$$\left. \cdot \int_0^{\pi/2}\left(\frac{1}{1+\frac{E_s^k}{N_0\sin^2(\theta)}}\right)^k d\theta\right]. \tag{14}$$

Thus, $\Delta = f(\xi_U) - f(\xi_L)$ can also be written as $\Delta = f(\xi_L + \Delta_o) - f(\xi_L)$. From the *Lemma* and (14) we can see that with the increases of E_s/N_0, both ξ_L and Δ_o become smaller. Let Δ_{ξ_L} and Δ_{Δ_o} represent the variations of ξ_L and Δ_o, respectively, with the changes of E_s/N_0. That is, if the transmitted symbol energy at a particular node is changed from E_{s1} to E_{s2}, then $\Delta_{\Delta_g} = \Delta_g(E_{s2}/N_0) - \Delta_g(E_{s1}/N_0)$, where $g \in \{\xi_L, o\}$. With the above notations, the ratio between Δ_{ξ_L} and Δ_{Δ_o} can be represented by

$$\frac{\Delta_{\xi_L}}{\Delta_{\Delta_o}} = \frac{\sum_{k=1}^{Q}\frac{1}{\pi}\binom{Q}{k}\beta_1^k(1-\beta_1)^{Q-k}}{\sum_{\kappa=2}^{K}\frac{\kappa}{K}\binom{K}{\kappa}\sum_{k=1}^{Q}\frac{1}{\pi}\binom{Q}{k}\beta_\kappa^k(1-\beta_\kappa)^{Q-k}} \tag{15}$$

It can be seen that E_s/N_0 is not included in (15). This means that no matter how the E_s/N_0 changes, the relationship between the variations Δ_{ξ_L} and Δ_{Δ_o} is always the same. Fig. 5 shows the ratio value changes as a function of the expanding

coefficient η. From Fig. 5 we can see that $\frac{\Delta_{\xi_L}}{\Delta_{\Delta_o}}$ is much smaller than 1 over all the expanding coefficients. It means that the variation of ξ_L is much smaller than that of Δ_o. Corresponding to this, $f(\xi_L + \Delta_o)$ changes much greater than $f(\xi_L)$ with the increase of E_s/N_0. Thus, the change of $\Delta = f(\xi_L + \Delta_o) - f(\xi_L)$ is mainly determined by $f(\xi_L + \Delta_o)$, that is to say, as E_s/N_0 increases, Δ becomes smaller.
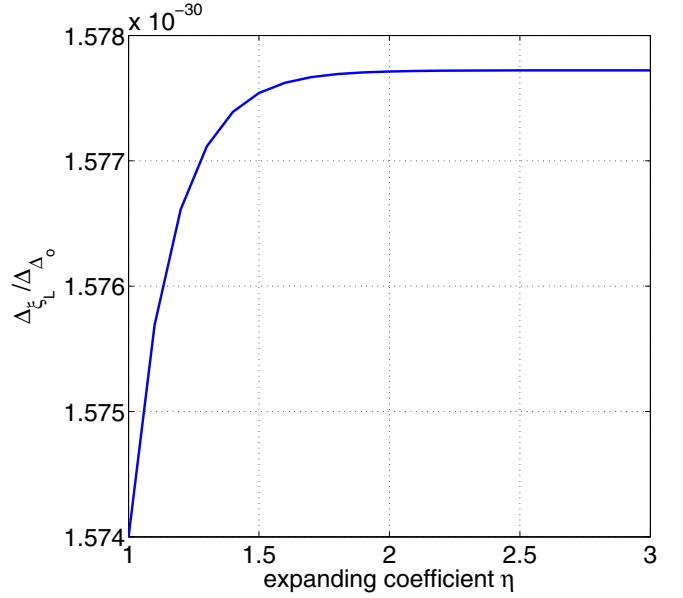

Fig. 5. The ratio $\frac{\Delta_{\xi_L}}{\Delta_{\Delta_o}}$.

From what has been discussed above, we may finally draw the conclusion that as E_s/N_0 increases, both $f(\xi_U)$ and Δ become smaller. Therefore, as a result of the increases of E_s/N_0, the difference between the upper bound ξ_U^{Raptor} of the proposed DNC scheme based on the Raptor codes and the lower bound ξ_L^{Raptor} becomes smaller.

The computational complexity of the ML decoding is $O(2^{\frac{K_L}{r}}Q)$, i.e., $O(2^K Q)$. The decoding complexity increases exponentially with the increasing of K_L. In order to obtain the simulation results with a reasonable decoding complexity, we consider the parameters $K_L = 18$, $K = 20$, $\Omega(x) = rx + (1-r)x^4$ and the degree distribution, $\mu(x) = 0.2x^2 + 0.35x^5 + 0.3x^7 + 0.115x^{10} + 0.01x^{12} + 0.025x^{14}$, which is designed in our other paper for simulation. Fig. 6 shows the BER performance bounds and simulation results of the constructed Raptor codes ensemble with $K = 20$ over Rayleigh fading channels under ML decoding at average E_s/N_0 of 7dB and 10dB, respectively. From the figure we can see that the simulation curve is between the upper and lower bounds of the constructed Raptor codes ensemble. As the upper and lower bounds are asymptotically tight as the expanding coefficient grows, the simulation curve coincident with the bounds, especially when the expanding coefficient is large enough. For example, $\eta = 2.5$ for $E_s/N_0 = 7$dB and $\eta = 2.0$ for $E_s/N_0 = 10$dB. Thus, we can come to the

conclusion that the derived analytical upper and lower bounds for the proposed Raptor-based DNC scheme are precise.
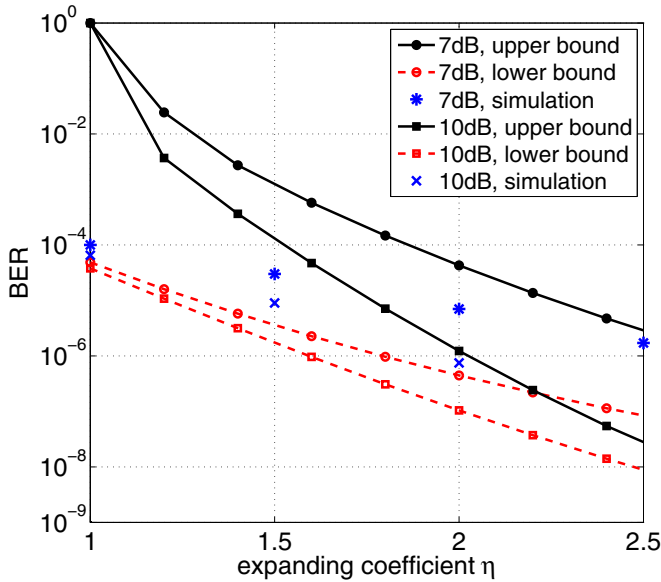

Fig. 6. BER performance bounds and simulation curves comparison for the constructed Raptor codes ensemble over Rayleigh fading channels with $K_L = 18$ and $K = 20$.

V. Conclusion

In this work, we first proposed a Raptor-based DNC scheme for WRNs. We proposed to use such a coding scheme for matching a code-on-graph to a network-on-graph to achieve a combined network-channel coding gain. Then, we derived analytical upper and lower bounds on ML decoding for the proposed Raptor-based DNC scheme. From the numerical and simulation results, we can see that the derived upper and lower BER bounds come closer, particularly as the length of the codewords is increased with a larger expanding coefficient. In our future work, we will use the developed bounds to optimize degree distributions of Fountain codes over Rayleigh fading channels.

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